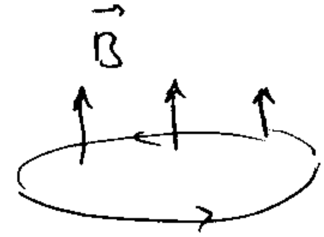
Circular accelerators:



$$p^0 = \frac{\mathcal{E}}{c} = \text{const}\ , \quad \frac{d\vec v}{dt} = \vec\omega \times \vec v$$

$$\Rightarrow \frac{d\vec\beta}{dt} = \vec\omega \times \vec\beta\ , \quad \vec\omega = \frac{e c \vec B}{\mathcal{E}} \quad \Rightarrow \dot{\vec\beta}^{\,2} = \omega^2 \beta^2$$

$$\vec\beta \times \dot{\vec\beta} = \vec\beta \times \left(\vec\omega \times \vec\beta\right) = \vec\omega\, \beta^2 - \vec\beta\, \cancelto{0}{\vec\omega \cdot \vec\beta}$$

$$\Rightarrow P = \frac{2}{3}\frac{e^2}{c}\gamma^6 \left[\omega^2\beta^2 - \omega^2\beta^4\right] = \frac{2}{3}\frac{e^2}{c}\gamma^4\beta^2\omega^2$$

the orbit's radius $r = \frac{v}{\omega} = \frac{\beta c}{\omega} \Rightarrow \omega = \frac{\beta c}{r}$

$$\Rightarrow P = \frac{2}{3}\frac{e^2}{\cancel{c}}\gamma^4\beta^2\, \frac{\beta^2 c^{\cancel{2}}}{r^2} \Rightarrow \frac{2}{3}\frac{e^2 c}{r^2}\beta^4\gamma^4 = P$$

Lienard 1898

$\Rightarrow$ energy loss per revolution is

$$\delta\mathcal{E} = P \cdot \frac{2\pi r}{v} = P\,\frac{2\pi r}{\beta c} = \frac{4\pi}{3}\frac{e^2}{r}\beta^3\gamma^4 \approx \Big|_{\beta \simeq 1}$$

$$\approx \frac{4\pi}{3}\frac{e^2}{r}\gamma^4 \approx \frac{4\pi}{3}\frac{e^2}{r}\left(\frac{E}{mc^2}\right)^4$$

at the same energy E electrons loose way more energy than protons: $\left(\frac{m_p}{m_e}\right)^4 \approx (2000)^4 \ldots$

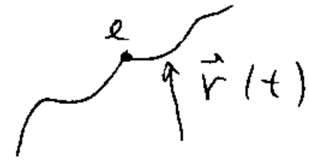
$$\delta\mathcal{E} \underset{\text{electrons}}{\simeq} 9\cdot 10^{-2}\,\frac{\left[\mathcal{E}(\text{GeV})\right]^4}{r\,(\text{meters})} \Rightarrow$$

LEP has $E = 60\,\text{GeV}$

$\Rightarrow \delta E = 300\,\text{MeV} \sim$ large...

Angular Distribution of Radiation.

Lienard - Wiechert field



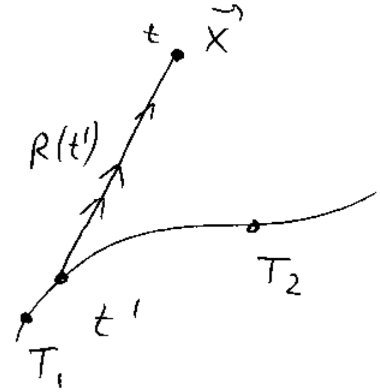
$$\vec{E}_{rad} = \frac{e}{c}\left[\frac{\hat{n}\times[(\hat{n}-\vec{\beta})\times\dot{\vec{\beta}}]}{(1-\vec{\beta}\cdot\hat{n})^3\, R}\right]_{ret}$$

$$\vec{B}_{rad} = \left[\hat{n}\times\vec{E}_{rad}\right]_{ret}$$

$$\vec{S} = \frac{c}{4\pi}\,\hat{n}\,|\vec{E}_{rad}|^2 \Rightarrow \left[\vec{S}\cdot\hat{n}\right]_{ret} = \frac{c}{4\pi}|\vec{E}_{rad}|^2 =$$

$$= \frac{e^2}{4\pi c}\left\{\frac{1}{R^2}\left|\frac{\hat{n}\times[(\hat{n}-\vec{\beta})\times\dot{\vec{\beta}}]}{(1-\hat{n}\cdot\vec{\beta})^3}\right|^2\right\}_{ret}$$

$\Rightarrow \left[\vec{S}\cdot\hat{n}\right]_{ret}$ is the power per unit area at point $\vec{x}$ and time t due to emission at time $t' = t - \frac{R(t')}{c}$:



$$\Rightarrow \left.\frac{\text{energy}}{\text{area}}\right|_{T_1<t'<T_2} = \int_{t=T_1+\frac{R(T_1)}{c}}^{t=T_2+\frac{R(T_2)}{c}} dt\cdot\left[\vec{S}\cdot\hat{n}\right]_{ret} =$$

$$= \int_{T_1}^{T_2} dt'\cdot\frac{dt}{dt'}\cdot\left[\vec{S}\cdot\hat{n}\right](t') \Rightarrow \text{as } \frac{dt_{ret}}{dt} = \left(\frac{1}{1-\hat{n}\cdot\vec{\beta}}\right)_{ret}$$

$$\Rightarrow \frac{dt}{dt'} = 1 - \hat{n}(t')\cdot\vec{\beta}\,(t')$$

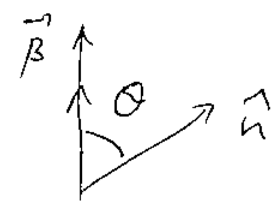
$\Rightarrow$ Power is $dP(t') = d\Omega \cdot R^2 \cdot \vec{S} \cdot \hat{n} \, \frac{dt}{dt'}$

$$\Rightarrow \boxed{\frac{dP(t')}{d\Omega} = \frac{e^2}{4\pi c} \frac{|\hat{n} \times [(\hat{n} - \vec{\beta}) \times \dot{\vec{\beta}}]|^2}{(1 - \hat{n} \cdot \vec{\beta})^5}}$$

Power per unit solid angle in terms of the charge's own time!

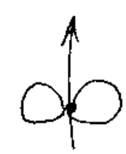
<u>Example</u>: linear motion $\vec{\beta} \parallel \dot{\vec{\beta}}$

$\hat{n} \cdot \vec{\beta} = \beta \cos\theta$



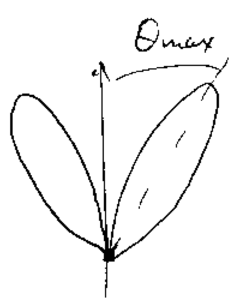
$$\boxed{\frac{dP(t')}{d\Omega} = \frac{e^2}{4\pi c} \frac{|\dot{\vec{\beta}}|^2 \sin^2\theta}{(1 - \beta\cos\theta)^5}} =$$

$$= \frac{e^2}{4\pi c^3} \frac{|\dot{\vec{v}}|^2 \sin^2\theta}{(1 - \beta\cos\theta)^5}$$



if $\beta \ll 1 \Rightarrow$ get $\frac{dP}{d\Omega} = \frac{e^2}{4\pi c^3} |\dot{\vec{v}}|^2 \sin^2\theta$ Larmor formula

if $\beta \sim 1 \Rightarrow$ get angular distribution like this



to find θ_{max} need to find the maximum of $\frac{\sin^2\theta}{(1 - \beta\cos\theta)^5}$

$$\frac{d}{d\theta}\frac{\sin^2\theta}{(1-\beta\cos\theta)^5} = \frac{2\sin\theta\cos\theta}{(1-\beta\cos\theta)^5} - 5\frac{\sin^2\theta\ \beta\cdot\sin\theta}{(1-\beta\cos\theta)^6} =$$

$$= \frac{\sin\theta}{(1-\beta\cos\theta)^6}\left[2\cos\theta(1-\beta\cos\theta) - 5\beta\sin^2\theta\right] = 0$$

$$\Rightarrow\ 2\cos\theta - 2\beta\cos^2\theta - 5\beta + 5\beta\cos^2\theta = 0$$

$$3\beta\cos^2\theta + 2\cos\theta - 5\beta = 0$$

$$\cos\theta_{1,2} = \frac{1}{3\beta}\left[-1 \pm \sqrt{1+15\beta^2}\right] \Rightarrow \cos\theta_{max} = \frac{\sqrt{1+15\beta^2}-1}{3\beta}$$

$\Rightarrow$ as $\beta^2 = 1 - \frac{1}{\gamma^2}$ $\Rightarrow$ in the UR limit $\gamma \to \infty$ $\Rightarrow$

$$\frac{1}{\gamma^2} \ll 1 \Rightarrow \beta \simeq 1 - \frac{1}{2\gamma^2} \Rightarrow \cos\theta_{max} \simeq \frac{\sqrt{16-\frac{15}{\gamma^2}}-1}{3\left(1-\frac{1}{2\gamma^2}\right)} \simeq$$

$$\simeq \left(4 - 1 - 4\frac{15}{32}\frac{1}{\gamma^2}\right)\frac{1}{3}\left(1+\frac{1}{2\gamma^2}\right) = \left(3 - \frac{15}{8}\frac{1}{\gamma^2}\right)\frac{1}{3}\cdot$$

$$\left(1+\frac{1}{2\gamma^2}\right) = 1 - \frac{5}{8}\frac{1}{\gamma^2} + \frac{1}{2\gamma^2} = 1 - \frac{1}{8}\frac{1}{\gamma^2}$$

$\Rightarrow$ if θ_{max} is small $\rightarrow$ $\cos\theta_{max} \simeq 1 - \frac{\theta_{max}^2}{2}$ $\Rightarrow$

$\Rightarrow$ $\theta_{max} \simeq \frac{1}{2\gamma}$ ~ tiny $\quad \left.\frac{dP}{d\Omega}\right|_{\theta=\theta_{max}} \simeq \frac{e^2}{4\pi c^3}\frac{\dot{v}^2\,\theta^2}{\left(1-\left(1-\frac{1}{\gamma^2}\right)\left(1-\frac{\theta^2}{2}\right)\right)^5}$

$$\simeq \frac{e^2}{4\pi c^3}\frac{\dot{v}^2\,\cancel{\theta}\left(\frac{1}{2\gamma}\right)^2}{\left(\frac{1}{\gamma^2}+\frac{1}{8\gamma^2}\right)^5} \propto \gamma^8 \text{ ~ large.}$$