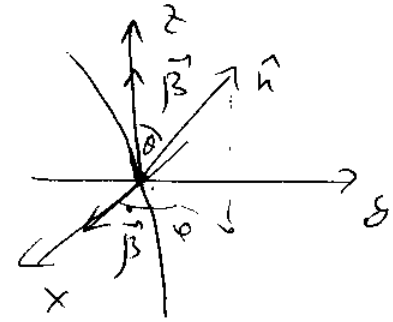
$$\frac{dP(t')}{d\Omega} \simeq \frac{e^2}{4\pi c^3} \frac{\dot{v}^2 \theta^2 \gamma^{10}}{\left(1+\frac{\gamma^2\theta^2}{2}\right)^5}, \quad \theta \ll 1.$$

Integrating the full expression over angles yields:

$$P = \frac{2}{3}\frac{e^2}{c^3}\dot{v}^2\gamma^6.$$

Example: circular motion

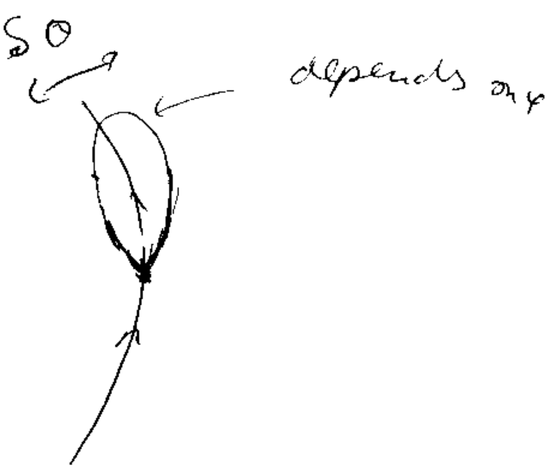


instantaneous picture

$\vec{\beta} \perp \dot{\vec{\beta}}$

$$\frac{dP(t')}{d\Omega} = \frac{e^2}{4\pi c^3} \frac{|\dot{\vec{v}}|^2}{(1-\beta\cos\theta)^3} \cdot$$

$$\cdot\left[1 - \frac{\sin^2\theta\cos^2\varphi}{\gamma^2(1-\beta\cos\theta)^2}\right]$$



again $\delta\theta \propto \frac{1}{\gamma}$

$$P_{total} = \frac{2}{3}\frac{e^2}{c^3}|\dot{\vec{v}}|^2\gamma^4.$$

Distribution in Frequency and Angle

$$\frac{dP(t)}{d\Omega} = \frac{c}{4\pi}\left[R^2\vec{E}^2(t_{ret})\right]_{ret} \equiv |\vec{A}(t)|^2$$

$$\Rightarrow \vec{A}(t) = \sqrt{\frac{c}{4\pi}}\left[R\vec{E}(t)\right]_{ret}$$

The total radiated energy is

$$\frac{dW}{d\Omega} = \int_{-\infty}^{\infty} dt \, \frac{dP}{d\Omega} = \int_{-\infty}^{\infty} dt \, |\vec{A}(t)|^2$$

$\Rightarrow$ Fourier decomposition is $\vec{A}(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} d\omega \, \vec{A}(\omega) e^{-i\omega t}$

$$\Rightarrow \int_{-\infty}^{\infty} dt \, |\vec{A}(t)|^2 = \frac{1}{2\pi} \int_{-\infty}^{\infty} dt \int_{-\infty}^{\infty} d\omega \, d\omega' \, \vec{A}(\omega) \, \vec{A}^*(\omega') \cdot e^{-it(\omega-\omega')}$$

$$= \int_{-\infty}^{\infty} d\omega \, |\vec{A}(\omega)|^2$$

as $\vec{A}(t)$ is real $\Rightarrow$ $\vec{A}^*(\omega) = \vec{A}(-\omega)$

$$\Rightarrow \frac{dW}{d\Omega} = \int_{-\infty}^{\infty} d\omega \, |\vec{A}(\omega)|^2 = \int_{0}^{\infty} d\omega \left(|\vec{A}(\omega)|^2 + |\vec{A}(-\omega)|^2 \right)$$

$$= \int_{0}^{\infty} d\omega \cdot 2 \, |\vec{A}(\omega)|^2 \equiv \int_{-\infty}^{\infty} d\omega \, \frac{dI}{d\omega \, d\Omega}$$

$$\Rightarrow \frac{dI}{d\omega \, d\Omega} = 2 \, |\vec{A}(\omega)|^2$$

as $\vec{A}(t) = \sqrt{\frac{c}{4\pi}} \left[R \vec{E} \right]_{ret} = \sqrt{\frac{c}{4\pi}} \, \frac{e}{c} \left[\frac{\hat{n} \times [(\hat{n} - \vec{\beta}) \times \dot{\vec{\beta}}]}{(1 - \vec{\beta} \cdot \hat{n})^3} \right]_{ret}$

$$\Rightarrow \vec{A}(\omega) = \left(\frac{e^2}{8\pi^2 c} \right)^{1/2} \int_{-\infty}^{\infty} dt \, e^{i\omega t} \left[\frac{\hat{n} \times [(\hat{n} - \vec{\beta}) \times \dot{\vec{\beta}}]}{(1 - \vec{\beta} \cdot \hat{n})^3} \right]_{ret}$$

Retarded time is defined by $t_{ret} = t - \frac{R(t_{ret})}{c}$.

$\Rightarrow t = t_{ret} + \frac{R(t_{ret})}{c} \Rightarrow$ as $\frac{dt}{dt_{ret}} = [1 - \hat{n}\cdot\vec{\beta}]_{ret} \Rightarrow$

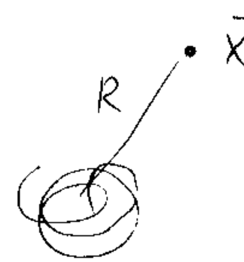
$$\vec{A}(\omega) = \left(\frac{e^2}{8\pi^2 c}\right)^{1/2} \int_{-\infty}^{\infty} dt' \, e^{i\omega\left(t' + \frac{R(t')}{c}\right)} \frac{\hat{n}\times[(\hat{n}-\vec{\beta})\times\dot{\vec{\beta}}]}{(1-\vec{\beta}\cdot\hat{n})^2}.$$

leading to

$$\frac{dI}{d\omega\, d\Omega} = \frac{e^2}{4\pi^2 c} \left| \int_{-\infty}^{\infty} dt' \, e^{i\omega\left(t' + \frac{R(t')}{c}\right)} \frac{\hat{n}\times[(\hat{n}-\vec{\beta})\times\dot{\vec{\beta}}]}{(1-\vec{\beta}\cdot\hat{n})^2} \right|^2$$

General result for radiation ω-spectrum & angular dependence.

$\Rightarrow$ assume that we are far from the charge, which is moving in a localized trajectory:



$R \gg |\vec{r}| \Rightarrow R = |\vec{x} - \vec{r}(t')| \simeq X - \hat{n}\cdot\vec{r}$

$$\Rightarrow \vec{A}(\omega) = \left(\frac{e^2}{8\pi^2 c}\right)^{1/2} \int_{-\infty}^{\infty} dt' \, e^{i\omega\left(t' + \frac{X}{c} - \frac{\hat{n}\cdot\vec{r}(t')}{c}\right)} \cdot \frac{\hat{n}\times[(\hat{n}-\vec{\beta})\times\dot{\vec{\beta}}]}{(1-\vec{\beta}\cdot\hat{n})^2}$$

$$\Rightarrow \frac{dI}{d\omega\, d\Omega} = \frac{e^2}{4\pi^2 c} \left| \int_{-\infty}^{\infty} dt' \, e^{i\omega\left(t' - \frac{\hat{n}\cdot\vec{r}(t')}{c}\right)} \cdot \frac{\hat{n}\times[(\hat{n}-\vec{\beta})\times\dot{\vec{\beta}}]}{(1-\vec{\beta}\cdot\hat{n})^2} \right|^2$$