Retarded time is defined by $t_{ret} = t - \frac{R(t_{ret})}{c}$.

$\Rightarrow t = t_{ret} + \frac{R(t_{ret})}{c} \Rightarrow$ as $\frac{dt}{dt_{ret}} = \left[1 - \hat{n}\cdot\vec{\beta}\right]_{ret} \Rightarrow$

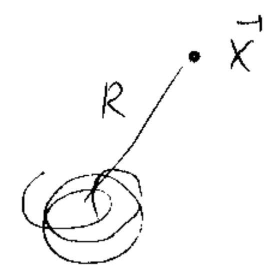
$$\vec{A}(\omega) = \left(\frac{e^2}{8\pi^2 c}\right)^{1/2} \int_{-\infty}^{\infty} dt'\, e^{i\omega\left(t' + \frac{R(t')}{c}\right)} \frac{\hat{n}\times[(\hat{n}-\vec{\beta})\times\dot{\vec{\beta}}]}{(1-\vec{\beta}\cdot\hat{n})^2}.$$

leading to

$$\frac{dI}{d\omega\, d\Omega} = \frac{e^2}{4\pi^2 c}\left|\int_{-\infty}^{\infty} dt'\, e^{i\omega\left(t' + \frac{R(t')}{c}\right)} \frac{\hat{n}\times[(\hat{n}-\vec{\beta})\times\dot{\vec{\beta}}]}{(1-\vec{\beta}\cdot\hat{n})^2}\right|^2$$

General result for radiation ω-spectrum & angular dependence.

$\Rightarrow$ assume that we are far from the charge, which is moving in a localized trajectory:



$R \gg |\vec{r}| \Rightarrow R = |\vec{x} - \vec{r}(t')| \simeq X - \hat{n}\cdot\vec{r}$

$$\Rightarrow \vec{A}(\omega) = \left(\frac{e^2}{8\pi^2 c}\right)^{1/2} \int_{-\infty}^{\infty} dt'\, e^{i\omega\left(t' + \frac{X}{c} - \frac{\hat{n}\cdot\vec{r}(t')}{c}\right)}.$$

$$\cdot \frac{\hat{n}\times[(\hat{n}-\vec{\beta})\times\dot{\vec{\beta}}]}{(1-\vec{\beta}\cdot\hat{n})^2} \Rightarrow \frac{dI}{d\omega\, d\Omega} = \frac{e^2}{4\pi^2 c}\left|\int_{-\infty}^{\infty} dt'\, e^{i\omega\left(t' - \frac{\hat{n}\cdot\vec{r}(t')}{c}\right)}.\right.$$

$$\left.\cdot \frac{\hat{n}\times[(\hat{n}-\vec{\beta})\times\dot{\vec{\beta}}]}{(1-\vec{\beta}\cdot\hat{n})^2}\right|^2$$

Now as $$\frac{\hat{n}\times[(\hat{n}-\vec{\beta})\times\dot{\vec{\beta}}]}{(1-\vec{\beta}\cdot\hat{n})^2} = \frac{d}{dt}\left[\frac{\hat{n}\times(\hat{n}\times\vec{\beta})}{(1-\vec{\beta}\cdot\hat{n})}\right]$$

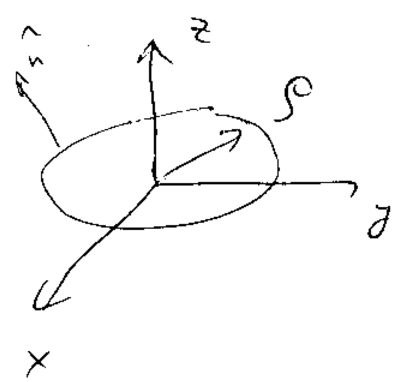
$\Rightarrow$ inserting and integrating by parts yields

$$\frac{dI}{d\omega\, d\Omega} = \frac{e^2}{4\pi^2 c}\left|\int_{-\infty}^{\infty} dt'\, e^{i\omega\left(t'-\frac{\hat{n}\cdot\vec{r}}{c}(t')\right)}\, i\omega\left(1-\hat{n}\cdot\vec{\beta}\right)\cdot\frac{\hat{n}\times(\hat{n}\times\vec{\beta})}{\cancel{1-\hat{n}\cdot\vec{\beta}}}\right|^2 = \frac{e^2\omega^2}{4\pi^2 c}\left|\int_{-\infty}^{\infty} dt'\, e^{i\omega\left(t'-\frac{\hat{n}\cdot\vec{r}(t')}{c}\right)}\cdot\hat{n}\times(\hat{n}\times\vec{\beta})\right|^2$$

$$\Rightarrow \frac{dI}{d\omega\, d\Omega} = \frac{e^2\omega^2}{4\pi^2 c}\left|\int_{-\infty}^{\infty} dt'\, e^{i\omega\left(t'-\frac{\hat{n}\cdot\vec{r}}{c}\right)}\,\hat{n}\times(\hat{n}\times\vec{\beta})\right|^2$$

Synchrotron Radiation

Imagine a particle in circular motion in xy-plane: $\vec{\beta} = \beta\left(\hat{x}\cos\left(\frac{Vt}{\rho}\right)+\hat{y}\sin\left(\frac{Vt}{\rho}\right)\right)$



ρ is the radius of the orbit,

V is particle's velocity

$\Rightarrow \hat{n}\times(\hat{n}\times\vec{\beta}) = \hat{n}(\hat{n}\cdot\vec{\beta})-\vec{\beta}$

if $\hat{n} = \sin\theta\cos\varphi\,\hat{x} + \sin\theta\sin\varphi\,\hat{y} + \cos\theta\,\hat{z} \Rightarrow$

$$\Rightarrow \hat{n}\cdot\vec{\beta} = \beta\sin\theta\left(\cos\varphi\cos\frac{vt}{\rho} + \sin\varphi\sin\frac{vt}{\rho}\right) =$$

$$= \beta\sin\theta\,\cos\left(\varphi - \frac{vt}{\rho}\right)$$

$$\Rightarrow \hat{n}\times(\hat{n}\times\vec{\beta}) = \hat{x}\left[\beta\sin^2\theta\cos\varphi\cos\left(\varphi-\frac{vt}{\rho}\right) - \beta\cos\frac{vt}{\rho}\right]$$

$$+\hat{y}\left[\beta\sin^2\theta\sin\varphi\cos\left(\varphi-\frac{vt}{\rho}\right) - \beta\sin\frac{vt}{\rho}\right] + \hat{z}\cos\theta\cdot\beta\sin\theta\cos\left(\varphi-\frac{vt}{\rho}\right)$$

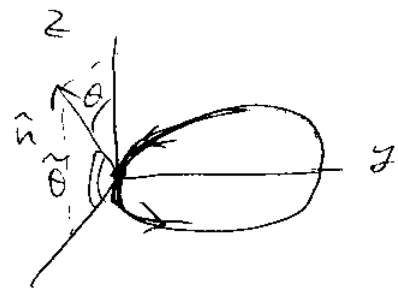
$$\cancel{+\beta\hat{y}\sin\left(\frac{vt}{\rho}\right) + \beta(\hat{n}\times\hat{y})\cos\left(\frac{vt}{\rho}\right)\sin\theta}$$

$$= \sin\frac{vt}{\rho}\left[\beta\hat{x}\sin^2\theta\cos\varphi\sin\varphi + \beta\hat{y}\left(\sin^2\theta\sin^2\varphi - 1\right)\right.$$

$$\left.+\hat{z}\beta\sin\theta\cos\theta\cdot\sin\varphi\right] + \cos\frac{vt}{\rho}\left[\hat{x}\left(\beta\sin^2\theta\cos^2\varphi - \beta\right)\right.$$

$$\left.+\hat{y}\,\beta\sin^2\theta\sin\varphi\cos\varphi + \hat{z}\,\beta\sin\theta\cos\varphi\cos\theta\right]$$

Imagine that at $t \simeq 0$ $\varphi = 0$:



$$\Rightarrow \hat{n}\times(\hat{n}\times\vec{\beta}) \simeq -\sin\left(\frac{vt}{\rho}\right)\beta\hat{y} +$$

$$+\cos\left(\frac{vt}{\rho}\right)\left[-\hat{x}\,\beta\cos^2\theta + \hat{z}\,\beta\sin\theta\cos\theta\right]$$

$$= -\sin\left(\frac{vt}{\rho}\right)\beta\underbrace{\hat{y}}_{=\hat{\varepsilon}_{\parallel}} + \cos\left(\frac{vt}{\rho}\right)\cdot\cos\theta\,\beta\underbrace{(\hat{n}\times\hat{y})}_{=\hat{\varepsilon}_{\perp}}$$

$$= \beta\left[-\sin\left(\frac{vt}{\rho}\right)\hat{\varepsilon}_{\parallel} + \cos\left(\frac{vt}{\rho}\right)\cos\theta\,\hat{\varepsilon}_{\perp}\right]$$

$$\omega\left(t' - \frac{\hat{n}\cdot\vec{r}}{c}\right) = \omega\left[t' - \frac{\rho}{c}\sin\theta\,\sin\left(\frac{vt}{\rho}\right)\right] \simeq$$

$$\simeq \Bigg|_{\substack{t\simeq 0 \\ \theta \simeq \frac{\pi}{2} - \tilde{\theta}}} = \omega\left[t - \frac{\rho}{c}\cos\tilde{\theta}\,\sin\frac{vt}{\rho}\right] \simeq$$

$$\simeq \omega\left[t - \frac{\rho}{c}\left(\frac{vt}{\rho} - \frac{1}{6}\left(\frac{vt}{\rho}\right)^3\right)\left(1 - \frac{\tilde{\theta}^2}{2}\right)\right] \simeq \omega\left[t\,(1-\beta) + \right.$$

$$\left. + \frac{\beta t\tilde{\theta}^2}{2} + \frac{v^3 t^3}{6c\rho^2}\right] \simeq \left| \begin{array}{l} \beta \to 1 \\ \Rightarrow \beta = \sqrt{\frac{-1}{\gamma^2} + 1} \simeq 1 - \frac{1}{2\gamma^2} \end{array}\right.$$

$$\simeq \omega\left[t\,\frac{1}{2\gamma^2} + \frac{t\,\tilde{\theta}^2}{2} + \frac{c^2 t^3}{6\rho^2}\right] \simeq \frac{\omega}{2}\left[t\left(\frac{1}{\gamma^2} + \tilde{\theta}^2\right) + \right.$$

$$\left. + \frac{c^2 t^3}{3\rho^2}\right].$$ We get

$$\frac{dI}{d\omega\, d\Omega} = \frac{e^2\omega^2}{4\pi^2 c}\left| -\hat{\varepsilon}_{\parallel} A_{\parallel}(\omega) + \hat{\varepsilon}_{\perp} A_{\perp}(\omega)\right|^2$$

with

$$A_{\parallel}(\omega) = \frac{c}{\rho}\int_{-\infty}^{\infty} dt'\cdot t'\cdot e^{i\frac{\omega}{2}\left[t'\left(\frac{1}{\gamma^2} + \tilde{\theta}^2\right) + \frac{c^2 t^3}{3\rho^2}\right]}$$

$$A_{\perp}(\omega) \simeq \tilde{\theta}\int_{-\infty}^{\infty} dt'\, e^{i\frac{\omega}{2}\left[t'\left(\frac{1}{\gamma^2} + \tilde{\theta}^2\right) + \frac{c^2 t^3}{3\rho^2}\right]}$$

A few more steps lead to