$$\boxed{\frac{dI}{d\omega\, d\Omega} = \frac{e^2}{3\pi^2 c}\left(\frac{\omega\rho}{c}\right)^2\left(\frac{1}{\gamma^2}+\theta^2\right)^2\left[K_{2/3}^{\,2}(\xi) + \frac{\theta^2}{\frac{1}{\gamma^2}+\theta^2}\cdot K_{1/3}^{\,2}(\xi)\right]}$$

where $\xi = \dfrac{\omega\rho}{3c}\left(\dfrac{1}{\gamma^2}+\theta^2\right)^{3/2}$

and $\theta = \tilde{\theta}$ (we dropped tilde)

modified Bessel functions $K_\nu(z)$ fall off exponentially at large $z \Rightarrow \xi \lesssim 1$

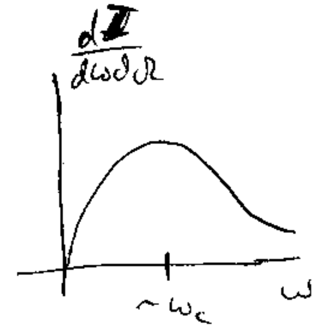
& radiation is suppressed for $\xi \gg 1$.

Look at $\theta = 0$: (BTW, the radiation cone's width is $\theta \sim \frac{1}{\gamma}$)

$\xi \lesssim 1 \Rightarrow \dfrac{\omega\rho}{3c\gamma^3} \lesssim 1 \Rightarrow$ the critical frequency

is
$$\boxed{\omega_c = \frac{3c\gamma^3}{\rho} = \frac{3c}{\rho}\left(\frac{E}{mc^2}\right)^3}$$

$$\frac{dI}{d\omega\, d\Omega} \propto \begin{cases} \omega^{2/3}, & \omega \ll \omega_c \\ \dfrac{\omega}{\omega_c}\, e^{-\omega/\omega_c}, & \omega \gg \omega_c \end{cases}$$



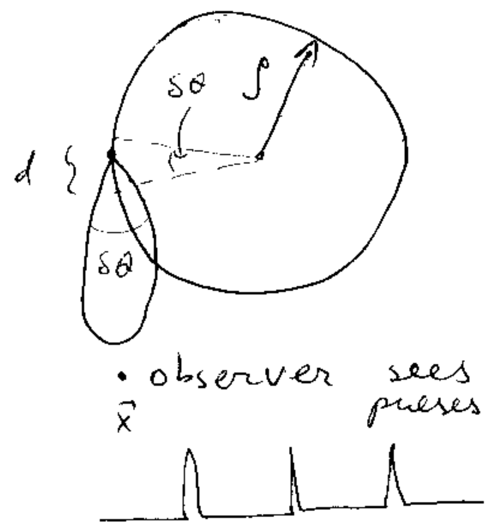
The radiation cone's width is

$$\theta_c \approx \frac{1}{\gamma}\left(\frac{2\omega_c}{\omega}\right)^{1/3} \quad \text{for } \omega \ll \omega_c$$

$$\theta_c \approx \frac{1}{\gamma}\left(\frac{2\omega_c}{3\omega}\right)^{1/2}, \quad \text{for } \omega \gg \omega_c \quad \text{narrower!}$$

Qualitative derivation of cutoff frequency:

$$d = \rho\,\delta\theta = \rho\,\frac{1}{\gamma} \quad \text{as } \delta\theta \sim \frac{1}{\gamma}$$



Front end of a pulse travels

distance $D = c\Delta t = c\,\dfrac{d}{v} =$

$$= \frac{d}{\beta} = \frac{\rho}{\beta\gamma}$$

Rear edge of a pulse is behind by ~~travels~~

$$L = D - d = \frac{\rho}{\beta\gamma} - \frac{\rho}{\gamma} = \frac{\rho}{\gamma}\left(\frac{1}{\beta} - 1\right) \approx \frac{\rho}{2\gamma^3}$$

$\Rightarrow$ the critical frequency is $\omega_c \approx \dfrac{c}{L} \sim \left(\dfrac{c}{\rho}\right)\gamma^3$ ~ correct estimate

"uncertainty principle"

as $k \cdot L \lesssim 1 \Rightarrow k = \frac{\omega}{c} \lesssim \frac{1}{L}$.

$\Rightarrow$ typically ω_c is in the x-ray range.

Radiation Reaction

If a particle radiates ⇒ it loses energy ⇒ her trajectory gets altered ⇒ in general have to re-think the problem. Luckily in majority of cases radiative energy loss is negligible.

$$P \sim \frac{e^2}{c^3}\dot{v}^2 \sim \frac{e^2}{c^3}a^2 \quad , \; a = \dot{v} \sim \text{typical acceleration}$$

$$E_{rad} = \int dt\, P = \frac{e^2}{c^3}a^2 T \quad , \; T \sim \text{period of motion}$$

The particle's kinetic energy $E_0 \sim m v^2 \sim m(aT)^2$

⇒ radiation is negligible if $\frac{e^2}{c^3}a^2 T \ll m(aT)^2$

(i.e. $E_{rad} \ll E_0$) $\Rightarrow \tau \simeq \frac{e^2}{mc^3} \ll T$

rad. reaction time

⇒ $T \gg 6.26 \cdot 10^{-24}\,s$ for electrons ⇒

⇒ radiative corrections are important mostly for elementary particles. ~ mostly quantum effects

$cT \gg 1\,fm$ ~ fermi, femto-meter (proton's radius)

Consider a non-relativistic particle:

$$m\dot{\vec{v}} = \vec{F}_{ext} + \vec{F}_{rad}$$

"usual" force — force due to radiation

Use Larmor formula $P(t) = \frac{2}{3}\frac{e^2}{c^3}\left(\dot{\vec{V}}\right)^2$

$\Rightarrow$ as $P(t) = -\vec{V}\cdot\vec{F}_{rad}$ $\Rightarrow$ the energy lost by a particle in some time interval $t_1 < t < t_2$ is:

$$\int_{t_1}^{t_2} dt\, P(t) = \underbrace{\frac{2}{3}\frac{e^2}{c^3}\int_{t_1}^{t_2} dt\,\left(\dot{\vec{V}}\right)^2}_{\text{parts}} = -\int_{t_1}^{t_2} dt\, \vec{V}\cdot\vec{F}_{rad}$$

$$\Rightarrow \int_{t_1}^{t_2} dt\, \vec{V}\cdot\vec{F}_{rad} = -\frac{2}{3}\frac{e^2}{c^3}\left\{\underbrace{\cancel{\vec{V}\cdot\dot{\vec{V}}\Big|_{t_1}^{t_2}}}_{0} - \int_{t_1}^{t_2} dt\, \vec{V}\cdot\ddot{\vec{V}}\right\}$$

0 (if motion is periodic or acceleration is only over a limited time, etc.)

$$\Rightarrow \boxed{\vec{F}_{rad} = \frac{2}{3}\frac{e^2}{c^3}\ddot{\vec{V}}}$$ radiative reaction force

And for the equation of motion we write

$$m\left(\dot{\vec{V}} - \tau\,\ddot{\vec{V}}\right) = \vec{F}_{ext}$$ Abraham - Lorentz equation of motion $\tau \equiv \frac{2e^2}{3mc^3}$

If $\vec{F}_{ext} = 0 \Rightarrow \dot{\vec{V}} = \tau\,\ddot{\vec{V}} \Rightarrow \vec{V} = \text{const}$ is a solution, but also $\vec{V} = \vec{V}_0\, e^{t/\tau}$:

$$\frac{d\dot{\vec{V}}}{dt} = \frac{\dot{\vec{V}}}{\tau} \Rightarrow 1d \Rightarrow \frac{d\dot{V}}{\dot{V}} = \frac{dt}{\tau} \Rightarrow$$

$$\Rightarrow \ln\dot{V} = \frac{t}{\tau} + \text{const}$$

$$\Rightarrow \dot{V} = a\cdot e^{t/\tau} \Rightarrow a \text{ may be } \neq 0.$$