Use Larmor formula $P(t) = \frac{2}{3}\frac{e^2}{c^3}\left(\dot{\vec{v}}\right)^2$

$\Rightarrow$ as $P(t) = -\vec{v}\cdot\vec{F}_{rad}$ $\Rightarrow$ the energy lost by a particle in some time interval $t_1 < t < t_2$ is:

$$\int_{t_1}^{t_2} dt\, P(t) = \frac{2}{3}\frac{e^2}{c^3}\underbrace{\int_{t_1}^{t_2} dt\,\left(\dot{\vec{v}}\right)^2}_{\text{parts}} = -\int_{t_1}^{t_2} dt\,\vec{v}\cdot\vec{F}_{rad}$$

$$\Rightarrow \int_{t_1}^{t_2} dt\,\vec{v}\cdot\vec{F}_{rad} = -\frac{2}{3}\frac{e^2}{c^3}\left\{\underbrace{\cancel{\vec{v}\cdot\dot{\vec{v}}}\Big|_{t_1}^{t_2}}_{0} - \int_{t_1}^{t_2} dt\,\vec{v}\cdot\ddot{\vec{v}}\right\}$$

(if motion is periodic or acceleration is only over a limited time, etc.)

$$\Rightarrow \boxed{\vec{F}_{rad} = \frac{2}{3}\frac{e^2}{c^3}\ddot{\vec{v}}}$$ radiative reaction force

And for the equation of motion we write

$$m\left(\dot{\vec{v}} - \tau\,\ddot{\vec{v}}\right) = \vec{F}_{ext}$$ Abraham - Lorentz equation of motion $\tau \equiv \frac{2e^2}{3mc^3}$

If $\vec{F}_{ext} = 0$ $\Rightarrow$ $\dot{\vec{v}} = \tau\,\ddot{\vec{v}}$ $\Rightarrow$ $\vec{v} = \text{const}$ is a solution, but also $\vec{v} = \vec{v}_0\, e^{t/\tau}$:

$$\frac{d\dot{\vec{v}}}{dt} = \frac{\dot{\vec{v}}}{\tau} \Rightarrow 1d \Rightarrow \frac{d\dot{v}}{\dot{v}} = \frac{dt}{\tau}$$

$$\Rightarrow \ln \dot{v} = \frac{t}{\tau} + \text{const}$$

$$\Rightarrow \dot{v} = a\cdot e^{t/\tau} \Rightarrow a \text{ may be } \neq 0.$$

In general higher order derivatives in EOM violate casuality!

Say, if $\vec{F}_{ext} \neq 0 \Rightarrow$ look for solution in the form $\dot{\vec{v}} = \vec{a}(t)\, e^{t/\tau} \Rightarrow -m\tau\, \dot{\vec{a}}\, e^{t/\tau} = \vec{F}_{ext}$

$$\Rightarrow \frac{d\vec{a}}{dt} = -\frac{1}{m\tau}\, e^{-t/\tau}\, \vec{F}_{ext}(t)$$

$$\Rightarrow \vec{a}(t) = -\frac{1}{m\tau} \int_{-\infty}^{t} dt'\, e^{-t'/\tau}\, \vec{F}_{ext}(t')$$

or

$$\vec{a}(t) = \frac{1}{m\tau} \int_{t}^{\infty} dt'\, e^{-t'/\tau}\, \vec{F}_{ext}(t')$$

$\Rightarrow$ get $\neq 0$ acceleration at $t = t_0$ even if $\vec{F}_{ext}(t_0) = 0$ but if $\vec{F}_{ext} \neq 0$ either at $t < t_0$ or $t > t_0$ $\Rightarrow$ say, the force in the future would affect acceleration at the present $\Rightarrow$

$\Rightarrow$ casuality violation.

$\Rightarrow$ Abraham - Lorentz equation is only valid if $\tau |\ddot{\vec{v}}| < |\dot{\vec{v}}| \Rightarrow \tau \ll T$,

i.e. when radiation effects are small.

<u>Example</u>: consider a harmonic oscillator

$$m\ddot{\vec{x}} = -k\vec{x} \Rightarrow \ddot{\vec{x}} = -\omega_0^2 \vec{x} \Rightarrow$$

$$\Rightarrow \vec{F}_{rad} = \frac{2}{3}\frac{e^2}{c^3}\dddot{\vec{x}} = -\frac{2}{3}\frac{e^2}{c^3}\omega_0^2 \dot{\vec{x}}$$

$$\Rightarrow m\ddot{\vec{x}} = -k\vec{x} - \underbrace{\frac{2}{3}\frac{e^2}{c^3}\omega_0^2}_{\equiv \gamma\cdot m} \dot{\vec{x}}$$

$$\Rightarrow \ddot{\vec{x}} + \gamma\dot{\vec{x}} + \omega_0^2\vec{x} = 0$$ ~ oscillator with friction

$$\gamma = \frac{2e^2\omega_0^2}{3mc^3} = \tau\omega_0^2$$ ~ "friction" or radiative friction.

$$\Rightarrow \ddot{\vec{x}} + \omega_0^2\tau\dot{\vec{x}} + \omega_0^2\vec{x} = 0$$

Solution is $\vec{x} = \vec{x}_0 e^{-i\omega t} \Rightarrow$

$$-\omega^2 - i\omega\tau\omega_0^2 + \omega_0^2 = 0 \Rightarrow \omega_{1,2} = \frac{-i\tau\omega_0^2}{2} \pm \sqrt{\omega_0^2 - \frac{\tau^2\omega_0^4}{4}}$$

$\Rightarrow$

$$\vec{x}(t) = \vec{x}_0 e^{-t\omega_0^2\tau/2} \cdot e^{\pm i\omega_0 t\sqrt{1-\frac{1}{4}\omega_0^2\tau^2}}$$

$\Gamma = \omega_0^2\tau$ ~ <u>decay constant</u>

as $\sqrt{1-\frac{1}{4}\omega_0^2\tau^2} \approx 1 - \frac{1}{8}\omega_0^2\tau^2 \Rightarrow \Delta\omega = -\frac{1}{8}\omega_0^3\tau^2$

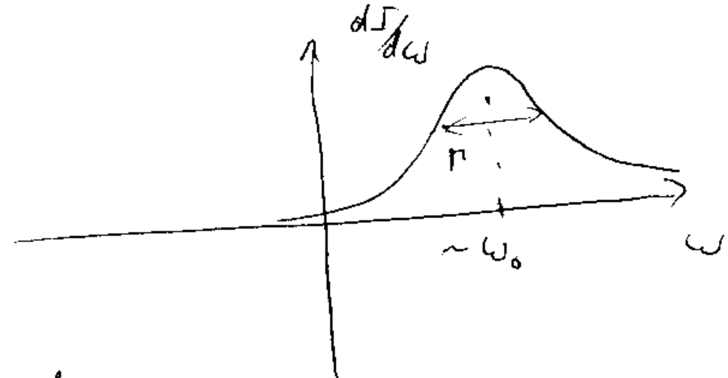
is the <u>level shift</u>.

$$\vec{X}(\omega) \propto \int_0^\infty dt \; e^{i\omega t} \; e^{-t\Gamma/2 - i\omega_0 t} \propto \frac{1}{\omega - \omega_0 + i\Gamma/2}$$

↖ start of oscillations

$$\Rightarrow \frac{dI}{d\omega} \propto |\vec{X}(\omega)|^2 \propto \frac{1}{(\omega-\omega_0)^2 + \Gamma^2/4}$$

Γ is a spectral line width (decay width)

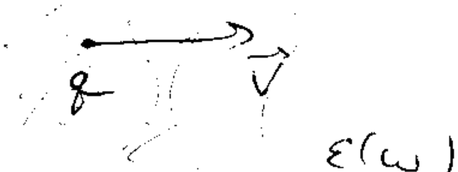


(e.g. consider an atom as an oscillator considered above $\Rightarrow$ Γ is a spectral line width, $\Delta\omega$ is a spectral level shift.)

Cherenkov Radiation and Energy Loss

Imagine a charge q moving with constant velocity $\vec{v}$ in a medium with dielectric function $\varepsilon(\omega)$

$$\begin{cases} \rho(\vec{x},t) = q\,\delta(\vec{x} - \vec{v}t) \\ \vec{J}(\vec{x},t) = q\vec{v}\,\delta(\vec{x} - \vec{v}t) \end{cases}$$



To find the electric field $\vec{E}$ (and magnetic field $\vec{B}$) due to this charge