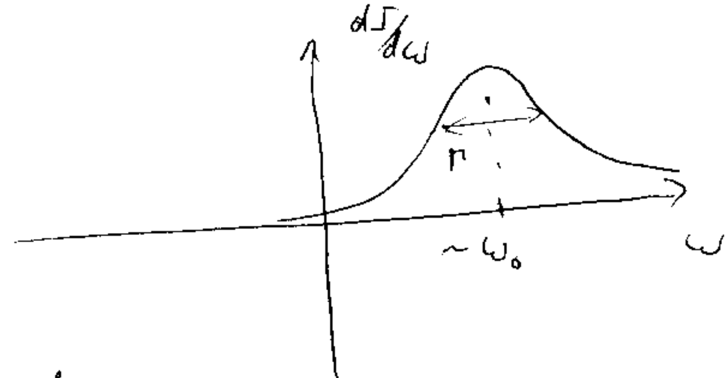
$$\vec{X}(\omega) \propto \int_0^\infty dt \; e^{i\omega t} \; e^{-t\Gamma/2 - i\omega_0 t} \propto \frac{1}{\omega - \omega_0 + i\Gamma/2}$$

↖ start of oscillations

$$\Rightarrow \frac{dI}{d\omega} \propto |\vec{X}(\omega)|^2 \propto \frac{1}{(\omega-\omega_0)^2 + \Gamma^2/4}$$

Γ is a spectral line width (decay width)

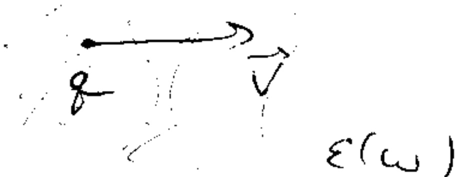


(e.g. consider an atom as an oscillator considered above $\Rightarrow$ Γ is a spectral line width, $\Delta\omega$ is a spectral level shift.)

Cherenkov Radiation and Energy Loss

Imagine a charge q moving with constant velocity $\vec{v}$ in a medium with dielectric function $\varepsilon(\omega)$

$$\begin{cases} \rho(\vec{x},t) = q\,\delta(\vec{x} - \vec{v}t) \\ \vec{J}(\vec{x},t) = q\vec{v}\,\delta(\vec{x} - \vec{v}t) \end{cases}$$



To find the electric field $\vec{E}$ (and magnetic field $\vec{B}$) due to this charge

We need to solve Maxwell equations in medium:

$$\begin{cases} \vec\nabla \times \vec E = -\frac{1}{c}\frac{\partial \vec B}{\partial t}, & \vec\nabla \times \vec B = \frac{4\pi}{c}\vec J + \frac{\varepsilon(\omega)}{c}\frac{\partial \vec E}{\partial t} \\ \vec\nabla \cdot \vec B = 0, & \varepsilon(\omega)\vec\nabla \cdot \vec E = 4\pi\rho \end{cases}$$

$\Rightarrow$ using $\vec E = -\vec\nabla \Phi - \frac{1}{c}\frac{\partial \vec A}{\partial t}$, $\vec B = \vec\nabla \times \vec A$ in Gauss's law yields:

$$-\varepsilon \nabla^2 \Phi - \frac{\varepsilon}{c}\frac{\partial}{\partial t}\vec\nabla\cdot\vec A = 4\pi\rho$$

in "Lorentz gauge" $\frac{\varepsilon}{c}\frac{\partial \Phi}{\partial t} + \vec\nabla\cdot\vec A = 0 \Rightarrow \vec\nabla\cdot\vec A = -\frac{\varepsilon}{c}\frac{\partial\Phi}{\partial t}$

$$\Rightarrow \quad -\varepsilon\nabla^2\Phi + \frac{\varepsilon^2}{c^2}\frac{\partial^2\Phi}{\partial t^2} = 4\pi\rho.$$

$$\frac{4\pi}{c}\vec J + \frac{\varepsilon(\omega)}{c}\frac{\partial \vec E}{\partial t} = \frac{4\pi}{c}\vec J - \frac{\varepsilon}{c}\vec\nabla\frac{\partial\Phi}{\partial t} - \frac{\varepsilon}{c^2}\frac{\partial^2\vec A}{\partial t^2} =$$

$$= \frac{4\pi}{c}\vec J + \cancel{\nabla^2\vec A}\;\vec\nabla(\vec\nabla\cdot\vec A) - \frac{\varepsilon}{c^2}\frac{\partial^2\vec A}{\partial t^2} = \vec\nabla\times\vec B =$$

$$= \vec\nabla\times(\vec\nabla\times\vec A) = \vec\nabla(\vec\nabla\cdot\vec A) - \nabla^2\vec A$$

$$\Rightarrow \quad -\nabla^2\vec A + \frac{\varepsilon}{c^2}\frac{\partial^2\vec A}{\partial t^2} = \frac{4\pi}{c}\vec J$$

Fourier-transforming $\vec A$, Φ:

$$\begin{pmatrix}\Phi \\ \vec A\end{pmatrix}(\vec x, t) = \frac{1}{(2\pi)^2}\int d^3k\, d\omega \begin{pmatrix}\Phi \\ \vec A\end{pmatrix}(\vec k, \omega)\, e^{i\vec k\cdot\vec x - i\omega t}$$

$\Rightarrow$ we get $\vec{\nabla}^2 \Rightarrow -\vec{k}^2$

the $\frac{\partial^2}{\partial t^2} \rightarrow -\omega^2$

$$\boxed{\begin{aligned} \left[\vec{k}^2 - \frac{\varepsilon(\omega)\omega^2}{c^2}\right] \Phi(\vec{k},\omega) &= \frac{4\pi}{\varepsilon(\omega)} \rho(\vec{k},\omega) \\ \left[\vec{k}^2 - \frac{\varepsilon(\omega)\omega^2}{c^2}\right] \vec{A}(\vec{k},\omega) &= \frac{4\pi}{c} \vec{J}(\vec{k},\omega) \end{aligned}}$$

where $\begin{pmatrix}\rho \\ \vec{J}\end{pmatrix}(\vec{k},\omega) = \int \frac{d^3x\, dt}{(2\pi)^2} e^{-i\vec{k}\cdot\vec{x} + i\omega t} \begin{pmatrix}\rho \\ \vec{J}\end{pmatrix}(\vec{x},t)$

$\Rightarrow$ for a point particle

$$\rho(\vec{k},\omega) = \int \frac{d^3x\, dt}{(2\pi)^2} e^{-i\vec{k}\cdot\vec{x} + i\omega t} q\, \delta(\vec{x} - \vec{v}t)$$

$$= \frac{q}{(2\pi)^2} \int_{-\infty}^{\infty} dt\, e^{-i\vec{k}\cdot\vec{v}t + i\omega t} = \frac{q}{2\pi} \delta(\omega - \vec{k}\cdot\vec{v})$$

$$\Rightarrow \boxed{\begin{aligned} \rho(\vec{k},\omega) &= \frac{q}{2\pi} \delta(\omega - \vec{k}\cdot\vec{v}) \\ \vec{J}(\vec{k},\omega) &= \frac{q}{2\pi}\vec{v}\, \delta(\omega - \vec{k}\cdot\vec{v}) \end{aligned}}$$

Solving the above equations for $\Phi, \vec{A}(\vec{k},\omega)$

we get

$$\Phi(\vec{k},\omega) = \frac{2q}{\varepsilon(\omega)} \frac{\delta(\omega - \vec{k}\cdot\vec{v})}{\vec{k}^2 - \frac{\varepsilon(\omega)\omega^2}{c^2}}$$

$$\vec{A}(\vec{k},\omega) = \frac{2q\vec{v}}{c} \frac{\delta(\omega - \vec{k}\cdot\vec{v})}{\vec{k}^2 - \frac{\varepsilon(\omega)\omega^2}{c^2}} = \varepsilon(\omega)\frac{\vec{v}}{c}\Phi(\vec{k},\omega)$$

To find electric field $\vec{E} = -\vec{\nabla}\Phi - \frac{1}{c}\frac{\partial \vec{A}}{\partial t}$ use

Fourier-space to write $\vec{E}(\vec{k},\omega) = -i\vec{k}\,\Phi(\vec{k},\vec{\omega}) +$

$+ i\frac{\omega}{c}\vec{A}(\vec{k},\omega) = i\left(\frac{\omega}{c}\frac{\vec{v}}{c}\varepsilon(\omega) - \vec{k}\right)\Phi(\vec{k},\omega).$

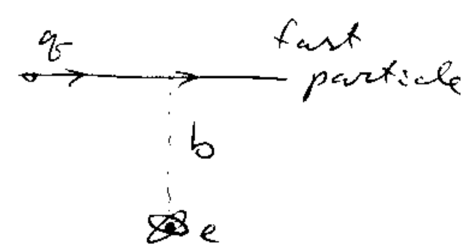
Also $\vec{B} = \vec{\nabla}\times\vec{A} \Rightarrow \vec{B}(\vec{k},\omega) = i\vec{k}\times\vec{A}(\vec{k},\omega) =$

$= i\,\varepsilon(\omega)\frac{1}{c}\left(\vec{k}\times\vec{v}\right)\Phi(\vec{k},\omega)$

$$\Rightarrow \quad \vec{E}(\vec{k},\omega) = i\left[\frac{\omega}{c}\frac{\vec{v}}{c}\varepsilon(\omega) - \vec{k}\right]\Phi(\vec{k},\omega)$$

$$\vec{B}(\vec{k},\omega) = i\,\varepsilon(\omega)\left(\vec{k}\times\frac{\vec{v}}{c}\right)\Phi(\vec{k},\omega)$$

Consider an electron in an atom at impact parameter b:



The fast particle looses energy due to work done on electrons in the medium. The force on the electron is $\vec{F} = e\vec{E} + \frac{e}{c}\hat{v} \times \vec{B}$ => work done is $\Delta E = -|e| \int_{-\infty}^{\infty} dt\ \vec{V}_e \cdot \vec{E}$ ≪ electric field of charge q

electron's velocity

Rewriting $\vec{V}_e(t) = \int_{-\infty}^{\infty} \frac{d\omega}{\sqrt{2\pi}} e^{-i\omega t} \vec{V}_e(\omega)$

$$\vec{E}(\vec{x},t) = \int_{-\infty}^{\infty} \frac{d\omega}{\sqrt{2\pi}} e^{-i\omega t} \vec{E}(\vec{x},\omega)$$

$$\Rightarrow \Delta E = -|e| \int_{-\infty}^{\infty} dt \int_{-\infty}^{\infty} \frac{d\omega\, d\omega'}{2\pi} e^{-i(\omega+\omega')t} \vec{V}_e(\omega) \cdot \vec{E}(\vec{x},\omega) = -|e| \int_{-\infty}^{\infty} d\omega\ \vec{V}_e(\omega) \cdot \vec{E}(\vec{x},-\omega)$$

Now, as $\vec{v}_e(t) = \frac{d\vec{x}}{dt} = -i\omega\vec{x}$, where $\vec{x}$ is electron's coordinate

$$\Rightarrow \Delta E = -|e|(-i) \int_{-\infty}^{\infty} d\omega\ \omega \cdot \vec{x} \cdot \vec{E}(\vec{x},-\omega) \underset{\text{is real}}{\overset{\text{as } \vec{E}(\vec{x},t)}{=}}$$

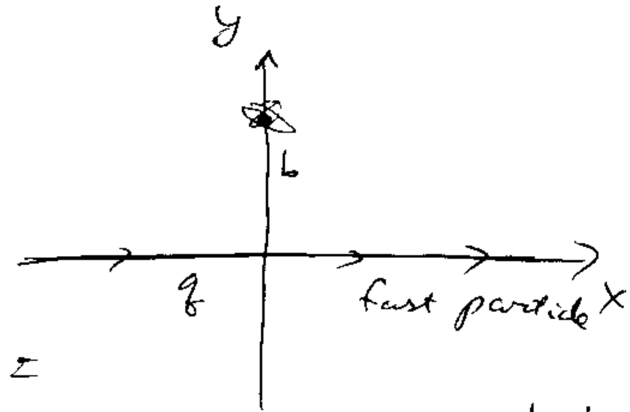
$$= |e| \int_{-\infty}^{\infty} d\omega\ i\omega\, \vec{x} \cdot \vec{E}^*(\vec{x},\omega) = \left(\text{like for } \frac{dI}{d\omega d\Omega}\right) =$$

$$= 2|e| \,\mathrm{Re} \int_{0}^{\infty} d\omega\ i\omega\, \vec{x} \cdot \vec{E}^*(\vec{x},\omega).$$

$$\Delta E = 2|e| \, \mathrm{Re} \int_0^\infty d\omega \; i\omega \, \vec{X}(\omega) \cdot \vec{E}^*(\vec{x},\omega)$$

$$\Rightarrow \text{as } \vec{E}(\vec{x},\omega) = \int \frac{d^3k}{(2\pi)^{3/2}} \vec{E}(\vec{k},\omega) \, e^{i\vec{k}\cdot\vec{x}}$$

position of electron, $(0,b,0)$



$$\Rightarrow \vec{k}\cdot\vec{x} = k_y b$$

$$\Rightarrow \vec{E}(\vec{x},\omega) = \int \frac{d^3k}{(2\pi)^{3/2}} \vec{E}(\vec{k},\omega) \, e^{ik_y b} =$$

$$= \int \frac{d^3k}{(2\pi)^{3/2}} e^{ik_y b} \cdot i \left[\frac{\omega}{c}\frac{\vec{v}}{c}\varepsilon(\omega) - \vec{k} \right] \frac{2q}{\varepsilon(\omega)} \frac{\delta(\omega - \overbrace{\vec{k}\cdot\vec{v}}^{k_x v})}{k^2 - \frac{\varepsilon\omega^2}{c^2}}$$

Start with E_x:

$$E_x(\vec{x},\omega) = \int \frac{d^3k}{(2\pi)^{3/2}} e^{ik_y b} \frac{2qi}{\varepsilon} \left[\frac{\omega}{c}\frac{v}{c}\varepsilon - k_x \right] \cdot$$

$$\cdot \frac{\delta(\omega - k_x v)}{k^2 - \frac{\varepsilon\omega^2}{c^2}} = \frac{2qi}{\varepsilon v} \int \frac{dk_y \, dk_z}{(2\pi)^{3/2}} e^{ik_y b} \left[\frac{\omega}{c}\frac{v}{c}\varepsilon - \frac{\omega}{v} \right] \cdot$$

$$\cdot \frac{1}{k_y^2 + k_z^2 - \varepsilon\frac{\omega^2}{c^2} + \frac{\omega^2}{v^2}}$$

Define $\quad \lambda^2 \equiv \frac{\omega^2}{v^2} - \varepsilon\frac{\omega^2}{c^2} = \frac{\omega^2}{v^2}\left[1 - \beta^2 \varepsilon(\omega)\right]$

which gives

$$E_x(\omega) = -\frac{2iq\omega}{V^2(2\pi)^{3/2}}\left[\frac{1}{\varepsilon(\omega)}-\beta^2\right]\int_{-\infty}^{\infty} dk_y\, dk_z\, e^{ik_y b} \cdot \frac{1}{k_y^2+k_z^2+\lambda^2}$$

The k_z-integral is $\displaystyle\int_{-\infty}^{\infty}\frac{dk_z}{k_z^2+k_y^2+\lambda^2} = 2\pi i\,\frac{1}{2i\sqrt{k_y^2+\lambda^2}} =$

$$= \frac{\pi}{\sqrt{k_y^2+\lambda^2}} \Rightarrow$$

$$E_x(\omega) = -\frac{iq\omega}{V^2(2\pi)^{1/2}}\left[\frac{1}{\varepsilon(\omega)}-\beta^2\right]\underbrace{\int_{-\infty}^{\infty} dk_y\, e^{ik_y b}\frac{1}{\sqrt{k_y^2+\lambda^2}}}_{2K_0(\lambda b)\ \ (\mathrm{Re}\,\lambda^2>0)}$$

$$\Rightarrow E_x(\omega) = -\frac{iq\omega}{V^2}\sqrt{\frac{2}{\pi}}\left[\frac{1}{\varepsilon(\omega)}-\beta^2\right]K_0(\lambda b)$$

Other non-zero components are

$$E_y(\omega) = \frac{q}{V}\sqrt{\frac{2}{\pi}}\,\frac{\lambda}{\varepsilon(\omega)}\,K_1(\lambda b)$$

$$B_z(\omega) = \varepsilon(\omega)\,\beta\,E_y(\omega)$$