Proper time & Time Dilation

Definition Proper time is the time in the rest frame of an object: $\boxed{d\tau \equiv \frac{ds}{c}}$.

Example: imagine a frame in which a particle moves with velocity $\vec{u}(t) \Rightarrow d\vec{x} = \vec{u}(t)\,dt$

$$\Rightarrow ds^2 = c^2 dt^2 - (d\vec{x})^2 = c^2 dt^2 - \vec{u}^2 dt^2 =$$

$$= c^2 dt^2 \left(1 - \beta^2(t)\right).$$

In the ~~sy~~ rest frame of the particle

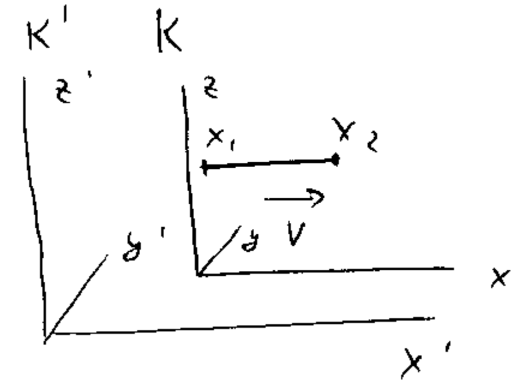
$$d\tau = \frac{ds}{c} = dt\,\sqrt{1-\beta^2(t)} \Rightarrow \boxed{\tau_2 - \tau_1 = \int_{t_1}^{t_2} dt\,\sqrt{1-\beta^2(t)}}.$$

alternatively

$$\boxed{t_2 - t_1 = \int_{\tau_1}^{\tau_2} \frac{d\tau}{\sqrt{1-\beta^2(\tau)}}} \Rightarrow \Delta t \geq \Delta\tau \text{ - time dilation.}$$

(E.g.: a photon has $\beta = 1 \Rightarrow \Delta\tau = 0 \Rightarrow$
$\Rightarrow$ the whole lifetime of the Universe is instanteneous for a photon!)

Lorentz Contraction.



Imagine a bar moving at constant velocity (see figure).

Proper length is defined as its length in the rest frame of the bar (frame K).

$$\Rightarrow x_1 = \frac{x_1' - vt'}{\sqrt{1-\frac{v^2}{c^2}}} \quad , \quad x_2 = \frac{x_2' - vt'}{\sqrt{1-\frac{v^2}{c^2}}}$$

$$\Rightarrow \Delta x = x_2 - x_1 = \frac{\Delta x'}{\sqrt{1-\frac{v^2}{c^2}}} = \frac{x_2' - x_1'}{\sqrt{1-\frac{v^2}{c^2}}}$$

$\Rightarrow$ if l_0 is proper length, $l_0 = \Delta x$

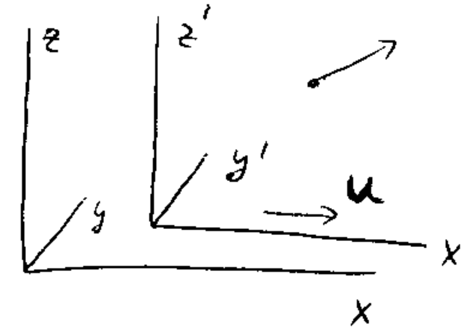
$$\Rightarrow l = \Delta x' \Rightarrow l = l_0 \sqrt{1-\frac{v^2}{c^2}}$$

$\Rightarrow l \leq l_0$ Lorentz contraction.

~ objects appear shorter in other frames.

Velocity Transformations.

$$\begin{cases} x = \gamma\,(x' + ut') \\ y = y' \\ z = z' \\ ct = (t'c + \beta x')\,\gamma \end{cases}$$



$$\Rightarrow V_x = \frac{dx}{dt} = \frac{dx' + u\,dt'}{dt' + \frac{\beta}{c}\,dx'} = \boxed{\frac{V_x' + u}{1 + \frac{uV_x'}{c^2}} = V_x}$$

where $V_x' = \dfrac{dx'}{dt'}$.

$$V_y = \frac{dy}{dt} = \frac{dy'}{\gamma\left(dt' + \frac{\beta}{c}\,dx'\right)} = \frac{V_y'}{\gamma\left(1 + \frac{\beta}{c}V_x'\right)} = \boxed{\frac{V_y'\sqrt{1 - \frac{u^2}{c^2}}}{1 + \frac{uV_x'}{c^2}} = V_y}$$

Similarly $\boxed{V_z = \dfrac{V_z'\sqrt{1 - \frac{u^2}{c^2}}}{1 + \frac{uV_x'}{c^2}}}$

($c \to \infty$ get Galilean $V_x = V_x' + u,\ V_y' = V_y,\ V_z' = V_z$)

Imagine the case when $V_x = V\cos\theta$, $V_y = V\sin\theta$, $V_z = 0$

$$\Rightarrow V_z' = 0, \quad \tan\theta = \frac{V_y}{V_x} = \frac{V_y'}{\gamma\,(V_x' + u)} \Rightarrow$$

$\Rightarrow$ write $V_x' = V'\cos\theta'$

$V_y' = V'\sin\theta'$

$$\Rightarrow \tan\theta = \frac{v'\sin\theta'}{\gamma(v'\cos\theta' + u)}$$

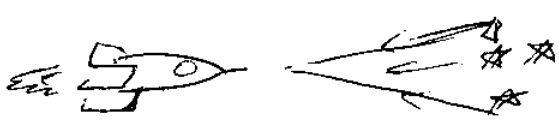
Light abberation: if the particle is a photon $\Rightarrow v = v' = c \Rightarrow \tan\theta = \dfrac{\sin\theta'}{\gamma(\cos\theta' + \beta)}$

$$\Rightarrow V_x = \frac{V_x' + u}{1 + \frac{u V_x'}{c^2}} \Rightarrow \cos\theta = \frac{\cos\theta' + \beta}{1 + \beta\cos\theta'}$$

$$V_y = \frac{V_y'}{\gamma\left(1 + \frac{u V_x'}{c^2}\right)} \Rightarrow \sin\theta = \frac{\sin\theta'}{\gamma(1 + \beta\cos\theta')}$$

$\Rightarrow$ if $u \to c \Rightarrow \sin\theta \to 0$, $\cos\theta \to 1 \Rightarrow$

$\Rightarrow \theta \to 0 \Rightarrow$ For UR spaceship all stars appear in the small cone ahead, at $\theta = 0$.



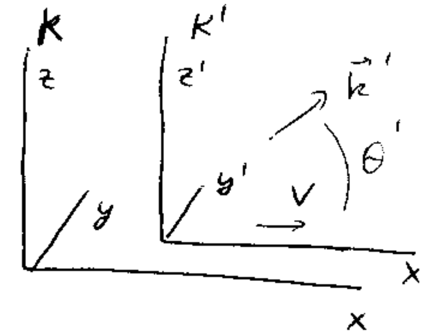
Four-vectors.

$x_0 = ct$, $x_1 = x$, $x_2 = y$, $x_3 = z$

(x_0, x_1, x_2, x_3) is a 4-vector.

Relativistic Doppler Shift.

Consider a plane wave with frequency ω & wave vector $\vec{k}$ in frame K. Assume that it has frequency ω' & wave vector $\vec{k}'$ in frame K'.



The phase of the wave is a number

$\Rightarrow$ invariant $\Rightarrow \varphi = \omega t - \vec{k}\cdot\vec{x} = \omega' t' - \vec{k}'\cdot\vec{x}'$

$\Rightarrow$ if the wave is moving in x/x' - direction ($\vec{k} \parallel \vec{V}$)

$\Rightarrow$ $k_y = k_y'$, $k_z = k_z'$

$x' = \gamma(x - Vt)$, $t' = \gamma(t - \frac{\beta}{c}x)$ $\Rightarrow$

$$\Rightarrow \underline{\omega t} - k_x \cdot x = \omega' \gamma\left(\underline{t} - \frac{\beta}{c}x\right) - k_x' \gamma(x - V\underline{t})$$

$$\left.\begin{aligned} \omega &= \gamma(\omega' + k_x' V) \\ k_x &= \gamma\left(\omega' \frac{\beta}{c} + k_x'\right) \end{aligned}\right\} \Rightarrow \begin{aligned} \omega' &= \gamma(\omega - V k_x) \\ k_x' &= \gamma\left(k_x - \beta \frac{\omega}{c}\right) \end{aligned}$$

For $\vec{k}$ pointing in any random direction: define $k_0 = \omega/c$, $k_0' = \omega'/c$ $\Rightarrow$ $\vec{\beta} \equiv \vec{V}/c$ $\Rightarrow$

$$\Rightarrow \begin{cases} k_0' = \gamma(k_0 - \vec{\beta}\cdot\vec{k}) \\ k_\parallel' = \gamma(k_\parallel - \beta k_0) \\ \vec{k}_\perp' = \vec{k}_\perp \end{cases}$$

where $k_\parallel$ is component $\parallel \vec{V}$, $\vec{k}_\perp$ are components $\perp \vec{V}$.

For an EM wave: $|\vec{k}| = k_0 = \frac{\omega}{c}$, $|\vec{k}'| = k_0' = \frac{\omega'}{c}$

$\Rightarrow$ if the angle between $\vec{k}$ and $\vec{v}$ is θ in K and θ' in K' $\Rightarrow$ $\omega' = \gamma(\omega - \beta\, \omega \cos\theta) \Rightarrow$

$$\Rightarrow \omega' = \gamma\omega(1-\beta\cos\theta)$$ Doppler shift

$$\begin{cases} \frac{\omega'}{c}\cos\theta' = \gamma\left(\frac{\omega}{c}\cos\theta - \beta\cdot\frac{\omega}{c}\right) \\ \frac{\omega'}{c}\sin\theta' = \frac{\omega}{c}\sin\theta \end{cases}$$

$$\Rightarrow \tan\theta' = \frac{\sin\theta}{\gamma(\cos\theta - \beta)}$$ (cf. with light abberation.)

Four - vectors.

We have seen one example: $x^0 = ct$, $x^1 = x$, $x^2 = y$, $x^3 = z$

$$\begin{pmatrix} x_0' \\ x_1' \\ x_2' \\ x_3' \end{pmatrix} = \begin{pmatrix} \gamma & -\beta\gamma & 0 & 0 \\ -\beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_0 \\ x_1 \\ x_2 \\ x_3 \end{pmatrix}$$