For an EM wave: $|\vec{k}| = k_0 = \frac{\omega}{c}$, $|\vec{k}'| = k_0' = \frac{\omega'}{c}$

$\Rightarrow$ if the angle between $\vec{k}$ and $\vec{v}$ is θ in K and θ' in K' $\Rightarrow$ $\omega' = \gamma(\omega - \beta\,\omega \cos\theta) \Rightarrow$

$$\Rightarrow \omega' = \gamma\omega(1 - \beta\cos\theta)$$ Doppler shift

$$\begin{cases} \frac{\omega'}{c}\cos\theta' = \gamma\left(\frac{\omega}{c}\cos\theta - \beta\cdot\frac{\omega}{c}\right) \\ \frac{\omega'}{c}\sin\theta' = \frac{\omega}{c}\sin\theta \end{cases}$$

$$\Rightarrow \tan\theta' = \frac{\sin\theta}{\gamma(\cos\theta - \beta)}$$ (cf. with light abberation.)

Four - vectors.

We have seen one example: $x^0 = ct$, $x^1 = x$, $x^2 = y$, $x^3 = z$

$$\begin{pmatrix} x_0' \\ x_1' \\ x_2' \\ x_3' \end{pmatrix} = \begin{pmatrix} \gamma & -\beta\gamma & 0 & 0 \\ -\beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_0 \\ x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

Definition A 4-vector A^μ is a set of 4 quantities (A^0, A^1, A^2, A^3), which under Lorentz transformation transform as

$$\begin{pmatrix} A^{0'} \\ A^{1'} \\ A^{2'} \\ A^{3'} \end{pmatrix} = \begin{pmatrix} \gamma & -\beta\gamma & 0 & 0 \\ -\beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} A^0 \\ A^1 \\ A^2 \\ A^3 \end{pmatrix}.$$

$\Rightarrow$ A^μ, $\mu = 0, 1, 2, 3$ is a <u>contravariant</u> vector if it transforms according to:

$$A'^\mu = \frac{\partial x'^\mu}{\partial x^\nu} A^\nu \quad \text{(equivalent to above)}$$

$\Rightarrow$ B_μ, $\mu = 0, \ldots, 3$ is a <u>covariant</u> vector if

$$B'_\mu = \frac{\partial x^\nu}{\partial x'^\mu} B_\nu$$

Example: $\frac{\partial \varphi}{\partial x^\mu}$ is a covariant vector as

$$\frac{\partial \varphi}{\partial x'^\mu} = \frac{\partial x^\nu}{\partial x'^\mu} \frac{\partial \varphi}{\partial x^\nu}.$$

One can define tensors by

$$A'^\mu B'^\nu = \frac{\partial x'^\mu}{\partial x^\alpha} \frac{\partial x'^\nu}{\partial x^\beta} \cdot A^\alpha A^\beta$$

$\Rightarrow$ rank two contravariant tensor would be $C'^{\mu\nu} = \frac{\partial x'^\mu}{\partial x^\alpha} \frac{\partial x'^\nu}{\partial x^\beta} C^{\alpha\beta}$, etc.

Definition: Scalar (inner) product of 2 vectors is defined by $A_\mu \cdot B^\mu$ (summation assumed)

Let's prove that it's Lorentz-invariant:

$$A'_\mu \cdot B'^\mu = \frac{\partial x^\alpha}{\partial x'^\mu} A_\alpha \frac{\partial x'^\mu}{\partial x^\beta} B^\beta = \frac{\partial x^\alpha}{\partial x^\beta} A_\alpha B^\beta = \delta^\alpha_\beta \cdot A_\alpha B^\beta = A_\alpha B^\alpha \quad \text{Q.E.D.}$$

The interval is a scalar: (it's Lorentz-invariant)

$$ds^2 = (dx^0)^2 - (dx^1)^2 - (dx^2)^2 - (dx^3)^2$$

Define the metric tensor by

$$ds^2 \equiv g_{\mu\nu}\, dx^\mu dx^\nu$$

$$g_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} = g^{\mu\nu} \quad \left(\text{Minkowski space}\right)$$

Note that $dx_\mu dx^\mu$ is also a Lorentz-scalar.

Identifying $dx_\mu = g_{\mu\nu} dx^\nu$ we see that

$g_{\mu\nu}$ raises and lowers the indices:

$$A_\mu = g_{\mu\nu} A^\nu \quad , \quad A^\mu = g^{\mu\nu} A_\nu$$

$(e.g. \quad x_\mu = g_{\mu\nu} x^\nu, \ldots)$

$$g^\mu{}_\nu = g^{\mu\alpha} \cdot g_{\alpha\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \delta^\mu{}_\nu$$

indeed, as $A_\mu B^\mu = \delta^\mu{}_\nu \cdot A_\mu B^\nu = g^\mu{}_\nu A_\mu B^\nu$.

Define an abbreviated notation: $\partial_\mu \equiv \dfrac{\partial}{\partial x^\mu}$

$$\partial^\mu \equiv \frac{\partial}{\partial x_\mu}$$

$\Rightarrow$ $\partial_\mu \varphi$ is a covariant vector

$\partial^\mu \varphi$ is a contravariant vector (check!)

$\partial_\mu A^\mu$ is Lorentz - invariant

Laplace operator $\dfrac{\partial^2}{c^2 \partial t^2} - \vec{\nabla}^2 = \partial_\mu \partial^\mu$ is

also Lorentz - invariant.

4 - velocity

Let's define a 4-vector for velocity:

$dx^\mu = (dx^0, dx^1, dx^2, dx^3) \quad \Rightarrow \quad V^\mu \stackrel{?}{=} \dfrac{dx^\mu}{dt}$?