$A_\mu = g_{\mu\nu} A^\nu \quad , \quad A^\mu = g^{\mu\nu} A_\nu$

(e.g. $x_\mu = g_{\mu\nu} x^\nu , \ldots$)

$$g^\mu{}_\nu = g^{\mu\alpha} \cdot g_{\alpha\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \delta^\mu{}_\nu$$

indeed, as $A_\mu B^\mu = \delta^\mu{}_\nu \cdot A_\mu B^\nu = g^\mu{}_\nu A_\mu B^\nu$.

Define an abbreviated notation: $\partial_\mu \equiv \frac{\partial}{\partial x^\mu}$

$$\partial^\mu \equiv \frac{\partial}{\partial x_\mu}$$

$\Rightarrow$ $\partial_\mu \varphi$ is a covariant vector

$\partial^\mu \varphi$ is a contravariant vector (check!)

$\partial_\mu A^\mu$ is Lorentz - invariant

Laplace operator $\frac{\partial^2}{c^2 \partial t^2} - \vec{\nabla}^2 = \partial_\mu \partial^\mu$ is

also Lorentz - invariant.

<u>4 - velocity</u>

Let's define a 4-vector for velocity:

$dx^\mu = (dx^0, dx^1, dx^2, dx^3) \Rightarrow v^\mu \stackrel{?}{=} \frac{dx^\mu}{dt}$?

But: time is not a <u>scalar</u> !

$\frac{dx^\mu}{dt} \sim \frac{dx^\mu}{dx^0} \sim$ not a Lorentz-vector.

$\Rightarrow$ try proper time $d\tau = \frac{ds}{c} \Rightarrow \boxed{u^\mu \equiv \frac{dx^\mu}{d\tau}}$ 4-velocity.

as $d\tau = \frac{dt}{\gamma} \Rightarrow u^0 = \frac{c\,dt}{dt/\gamma} = c\gamma$

$\vec{u} = \frac{d\vec{x}}{dt} \cdot \gamma = \gamma \cdot \vec{v} \Rightarrow \boxed{u^\mu = \gamma(c, \vec{v})}$.

Note $u_\mu u^\mu = c^2$.

<u>Boost in terms of rapidity</u>

$$\begin{pmatrix} A'^0 \\ A'^1 \\ A'^2 \\ A'^3 \end{pmatrix} = \begin{pmatrix} \gamma & -\beta\gamma & 0 & 0 \\ -\beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} A^0 \\ A^1 \\ A^2 \\ A^3 \end{pmatrix}$$

$\Rightarrow$ Define rapidity η

by $\boxed{\beta \equiv \tanh \eta} = \frac{e^\eta - e^{-\eta}}{e^\eta + e^{-\eta}}$

Define <u>light-cone coordinates</u> $A^+ = \frac{A^0 + A^1}{\sqrt{2}}$

$A^- = \frac{A^0 - A^1}{\sqrt{2}}$

$\Rightarrow$ then

$A_\mu A^\mu = 2A^+A^- - (A^2)^2 - (A^3)^2$

$$A'^{+} = \frac{1}{\sqrt{2}}\left(A'^{0} + A'^{1}\right) = \frac{1}{\sqrt{2}}\,\gamma\left(A^{0} - \beta A^{1} - \beta A^{0} + A^{1}\right) =$$

$$= \frac{1}{\sqrt{2}}\,\gamma\,(1-\beta)\left(A^{0} + A^{1}\right) = \gamma\,(1-\beta)\,A^{+}$$

$$A'^{-} = \frac{1}{\sqrt{2}}\left(A'^{0} - A'^{1}\right) = \gamma\,(1+\beta)\,A^{-}$$

$\gamma(1-\beta)\cdot\gamma(1+\beta) = 1 \Rightarrow$ define $\gamma(1-\beta) = e^{-\eta} \Rightarrow$

$\gamma(1+\beta) = e^{+\eta} \Rightarrow \dfrac{1-\beta}{1+\beta} = e^{-2\eta} \Rightarrow \dfrac{1-e^{-2\eta}}{1+e^{-2\eta}} = \beta \Rightarrow$

$\Rightarrow \beta = \tanh\eta$.

$\Rightarrow$ Rapidity makes boosts easy!

$$A'^{+} = e^{-\eta} A^{+} \;;\quad A'^{-} = e^{\eta} A^{-} \;;\quad A'^{2,3} = A^{2,3}.$$

$$-\infty < \eta < +\infty.$$

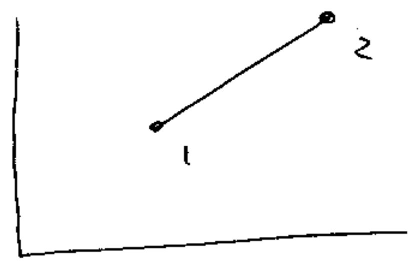
Two boosts: $A''^{+} = e^{-\eta_2} A'^{+} = e^{-\eta_1-\eta_2} A^{+}$

$$A''^{-} = e^{\eta_1+\eta_2} A^{-}$$

$\sim$ a simple addition!

Relativistic Mechanics

Consider a free particle (moving along a straight line). We need to construct a Lorentz-invariant action for such particle. It's characterized by a 4-vector x^μ $\Rightarrow$ the only Lorentz-invariant is the interval $\Rightarrow$



$$\Rightarrow \int_1^2 ds \qquad \left(\text{can't have } \int_1^2 (ds)^2 \sim \text{still infinitesimal}\right)$$

$\Rightarrow$ write action as $S = -A \cdot \int_1^2 ds$

As $ds^2 = c^2dt^2 - (d\vec{x})^2 = dt^2\left(1-\beta^2(t)\right) \Rightarrow$

$\Rightarrow ds = c\,dt\sqrt{1-\beta^2(t)} \Rightarrow S = -Ac\int_{t_1}^{t_2} dt\sqrt{1-\beta^2(t)}$.

$\Rightarrow$ as $S = \int_{t_1}^{t_2} dt \cdot L$, where L is the Lagrangian.

$\Rightarrow L = -Ac\sqrt{1-\beta^2(t)}$.

Now, in classical non-relativistic mechanics we know that $L = T - V$

kinetic energy (T), potential energy (V)

$\Rightarrow$ for a free NR particle $V = 0 \Rightarrow L = T = \frac{1}{2}mv^2$

(We know that in non-relativistic (NR) limit: $T = \frac{1}{2}mv^2$) $\Rightarrow$ as $\beta \to 0 \Rightarrow$

$$\Rightarrow L = \underbrace{-Ac}_{\text{constant} \sim \text{drop, not important for dynamics}} + Ac\frac{1}{2}\beta^2 + \dots$$

$$\Rightarrow Ac\frac{1}{2}\frac{v^2}{c^2} = \frac{1}{2}mv^2 \Rightarrow A = mc$$

$$\Rightarrow \boxed{S = -mc\int_1^2 ds = -mc^2\int_{t_1}^{t_2} dt\sqrt{1-\beta^2(t)}}$$

$$\boxed{L = -mc^2\sqrt{1-\beta^2(t)}}$$

The particle's Energy & Momentum.

The free particle's degrees of freedom are coordinates $\vec{x}$ & t. Momentum is defined by: $p^i = \dfrac{\partial L}{\partial \dot{x}_i}$, where $i = 1, 2, 3$ and $\dot{x}^i = \dfrac{dx^i}{dt}$.

(know from classical mechanics).

$$\Rightarrow \; p^i = \frac{\partial L}{\partial v_i} = -mc^2 \frac{-\cancel{2}v^i/c^2}{\cancel{2}\sqrt{1-\frac{v^2}{c^2}}}$$

$$\Rightarrow \; \vec{p} = \frac{m\vec{v}}{\sqrt{1-\frac{v^2}{c^2}}} = \gamma m \vec{v}$$

Energy is defined by $E = \vec{p}\cdot\dot{\vec{x}} - L =$

$$= \vec{p}\cdot\vec{v} - L = \gamma m v^2 + mc^2\sqrt{1-\frac{v^2}{c^2}} =$$

$$= \gamma\left[\cancel{mv^2} + mc^2\left(1 - \cancel{\frac{v^2}{c^2}}\right)\right] = mc^2\gamma$$

$$\Rightarrow \; E = mc^2\gamma = \frac{mc^2}{\sqrt{1-\frac{v^2}{c^2}}}.$$