Remember that $u^\mu = \gamma(c, \vec{v})$ = 4-velocity.

We now see that $\left(\frac{E}{c}, \vec{p}\right) = m\gamma(c, \vec{v}) \Rightarrow$

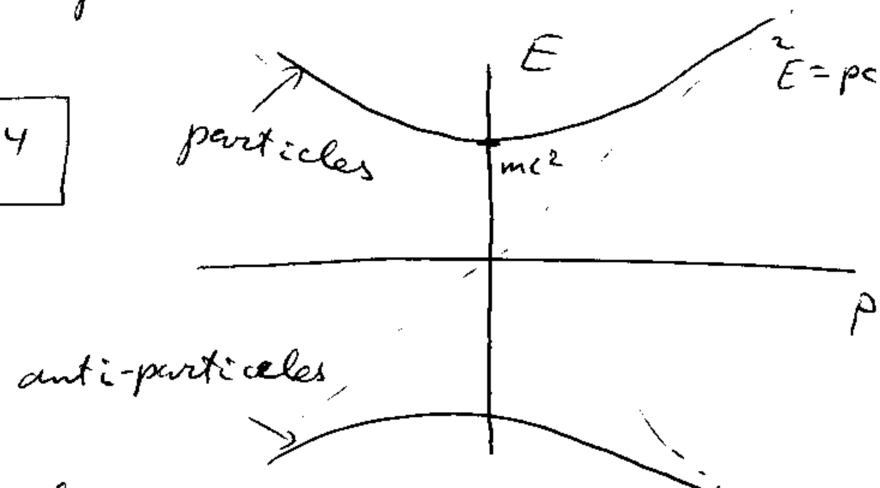
$\Rightarrow$ we have a new <u>4-momentum</u> four-vector:

$\boxed{p^\mu = m u^\mu}$, where $p^0 = \frac{E}{c}$, $p^i = (\vec{p})^i$

Note that $p_\mu p^\mu = m^2 u_\mu u^\mu = m^2\gamma^2(c^2 - v^2) =$

$= m^2c^2 \Rightarrow \frac{E^2}{c^2} - \vec{p}^{\,2} = m^2c^2$ or

$$\boxed{E^2 = \vec{p}^{\,2}c^2 + m^2c^4}$$



4-vector p^μ transforms in the usual way:

$$\boxed{\begin{aligned} p_x &= \gamma(p_x' + \beta p_0') \\ p_0 &= \gamma(p_0' + \beta p_x') \\ p_y &= p_y', \quad p_z = p_z' \end{aligned}}$$

Def.

<u>kinetic energy</u>

$T = E(v) - E(0) =$
$= mc^2[\gamma_u - 1]$

Newtonian mechanics: $\vec{F} = \frac{d\vec{p}}{dt}$ (force)

$\Rightarrow$ define force as $f^\mu = \frac{dp^\mu}{d\tau}$

$\Rightarrow \vec{f} = \frac{d\vec{p}}{dt}\gamma \Rightarrow$ in NR limit gives Newtonian result.
"$\vec{F}\cdot\gamma$

$\frac{dp^0}{d\tau} = \gamma \frac{dp^0}{dt}$; Note that $f^\mu u_\mu = 0$

$$\left(u_\mu f^\mu = u_\mu \frac{dp^\mu}{d\tau} = u_\mu m \frac{du^\mu}{d\tau} = \frac{1}{2} m \frac{d(u_\mu u^\mu)}{d\tau} = \right.$$

$$\left. = \frac{1}{2} m \frac{dc^2}{d\tau} = 0 \right) \Rightarrow f^0 \cdot u^0 = \vec{f}\cdot \gamma \vec{v} \Rightarrow$$

$$\Rightarrow f^0 c = \vec{f}\cdot\vec{v} \Rightarrow f^0 = \frac{\vec{f}\cdot\vec{v}}{c} \Rightarrow \gamma \frac{dp^0}{dt} = f^0 = \frac{\vec{f}\cdot\vec{v}}{c}$$

$$\Rightarrow \gamma \frac{dE}{dt} = \vec{f}\cdot\vec{v} = \gamma \vec{F}\cdot\vec{v} \Rightarrow \frac{dE}{dt} = \vec{F}\cdot\vec{v}$$

($\vec{F}$ is Newtonian NR force).

$\Rightarrow$ 4-momentum is conserved in particle interactions.

$$\sum p^\mu_{initial} = \sum p^\mu_{final}$$

Particle Decay

Imagine a particle with mass M at rest which decays into 2 particles with masses m_1 & m_2:

$$\overset{\bullet}{M} \;\rightarrow\; \overleftarrow{m_1}\;\; \overrightarrow{m_2}$$

$$p^\mu = p_1^\mu + p_2^\mu \,, \text{ where } p^\mu = (Mc, \vec{0})$$

$$p_1^\mu = \left(\frac{E_1}{c}, \vec{p}_1\right), \quad p_2^\mu = \left(\frac{E_2}{c}, \vec{p}_2\right)$$

$\Rightarrow \mu = 0 \Rightarrow$ energy conservation $\Rightarrow$

$$mc = \frac{E_1}{c} + \frac{E_2}{c}$$

$\mu = i \Rightarrow$ momentum conservation: $\vec{p}_1 + \vec{p}_2 = 0$

Rewrite $p^\mu - p_1^\mu = p_2^\mu \Rightarrow$ square $\Rightarrow$

$$(p - p_1)^2 = p_2^2 = m_2^2 c^2 \Rightarrow p^2 + p_1^2 - 2p\cdot p_1 = m_2^2 c^2$$

$$\Rightarrow M^2c^2 + m_1^2c^2 - 2Mc\cdot\frac{E_1}{c} = m_2^2 c^2$$

$$\Rightarrow E_1 = \frac{M^2 + m_1^2 - m_2^2}{2M}\, c^2$$

$\Rightarrow$ as $E_1 > m_1 c^2$

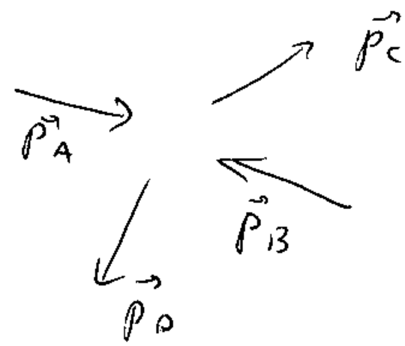
$\Rightarrow M > m_1 + m_2$

otherwise decay can't happen.

similarly
$$E_2 = \frac{M^2 + m_2^2 - m_1^2}{2M}$$

Particle Scattering

Imagine particles A & B colliding to become particles B & C:



$$p_A^\mu + p_B^\mu = p_C^\mu + p_D^\mu .$$

$$\Rightarrow \quad \vec{p}_A + \vec{p}_B = \vec{p}_C + \vec{p}_D$$

$$E_A + E_B = E_C + E_D$$

<u>Invariant mass</u> of the particles: $M^2 = \left(\sum_i p_i^\mu\right)^2$

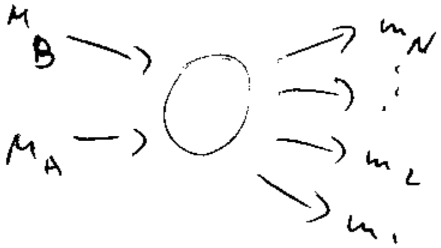
e.g. $M^2 = (p_A + p_B)^2$.

also known as ceter-of-mass energy : $s = (p_A + p_B)^2$.

(Go to the center-of-mass frame where

$\vec{p}_A{}' + \vec{p}_B{}' = 0 \Rightarrow (p_A + p_B)^2 = \left(\frac{E_1'}{c} + \frac{E_2'}{c}\right)^2$ ← only energy in CMS frame.)

<u>Threshold energy</u>:



$$\Rightarrow (p_A + p_B)^2 = s = (p_1 + \ldots + p_N)^2 =$$

$$= (E_1' + \ldots + E_N')^2 \frac{1}{c^2} \geq \frac{(m_1 + \ldots + m_N)^2}{c^2}$$

↑ CMS frame

⇒ minimum energy for a process to take place is $\boxed{S \geq S_{min} = \frac{1}{c^2}(m_1 + \dots + m_N)^2}$.

Threshold energy.

Covariance of Electrodynamics

Start with Maxwell equations in vacuum:
(microscopic Maxwell eqns)

$$\begin{cases} \vec{\nabla}\cdot\vec{E} = 4\pi\rho & \vec{\nabla}\cdot\vec{B} = 0 \\ \vec{\nabla}\times\vec{E} = -\frac{1}{c}\frac{\partial\vec{B}}{\partial t} & \vec{\nabla}\times\vec{B} = \frac{4\pi}{c}\vec{J} + \frac{1}{c}\frac{\partial\vec{E}}{\partial t} \end{cases}$$

(in ~~SI~~ Gaussian units now, in vacuum $\vec{B} = \vec{H}$, $\vec{E} = \vec{D}$).

$\left(\vec{D} = \vec{E} + 4\pi\vec{P},\ \ \vec{H} = \vec{B} - 4\pi\vec{M} \text{ in general}\right)$

Define scalar and vector potentials by

$$\vec{E} = -\vec{\nabla}\Phi - \frac{1}{c}\frac{\partial\vec{A}}{\partial t}, \quad \vec{B} = -\vec{\nabla}\times\vec{A}.$$

Pick Lorenz gauge condition: $\frac{1}{c}\frac{\partial\Phi}{\partial t} + \vec{\nabla}\cdot\vec{A} = 0$

$\Rightarrow$ Gauss's law gives: $-\nabla^2\Phi - \frac{1}{c}\frac{\partial}{\partial t}\vec{\nabla}\cdot\vec{A} = 4\pi\rho$

$$\Rightarrow \boxed{\frac{1}{c^2}\frac{\partial^2\Phi}{\partial t^2} - \nabla^2\Phi = 4\pi\rho}.$$

Ampere's law leads to: $\vec{\nabla}(\vec{\nabla}\cdot\vec{A}) - \nabla^2\vec{A} = \frac{4\pi}{c}\vec{J} -$

$$\cancel{-\frac{1}{c}\vec{\nabla}\frac{\partial\Phi}{\partial t}} - \frac{1}{c^2}\frac{\partial^2\vec{A}}{\partial t^2} \Rightarrow \boxed{\frac{1}{c^2}\frac{\partial^2\vec{A}}{\partial t^2} - \nabla^2\vec{A} = \frac{4\pi}{c}\vec{J}}.$$

Before we continue, let's establish transformation properties of $\vec{J}, \rho$: we know that $\frac{\partial \rho}{\partial t} + \vec{\nabla} \cdot \vec{J} = 0$ (continuity) $\Rightarrow$ defining an object $J^\mu \equiv (c\rho, \vec{J})$

we write the condition as $\boxed{\partial_\mu J^\mu = 0}$

$\partial_\mu \sim$ covariant 4-vector

If continuity holds in all frames $\Rightarrow \partial_\mu J^\mu$ is Lorentz-scalar $\Rightarrow J^\mu$ is a contravariant 4-vector!

<u>Charge conservation</u>: the charge in a volume element d^3x is the same in any frame:

in some frame K have $\rho(\vec{x}, t)\, d^3x$

in another frame K' have $\rho'(\vec{x}', t')\, d^3x'$

$\Rightarrow \rho\, d^3x = \rho'\, d^3x'$

Is this correct?

$\Rightarrow$ Note that $d^4x = c\,dt\, d^3x = dx^0 d^3x$ is a Lorentz-scalar: take a general Lorentz-transform, $x'^\mu = \Lambda^\mu{}_\nu x^\nu \Rightarrow d^4x' = d^4x \cdot \det\Lambda$

Jacobian

$\Rightarrow \det \Lambda = \det \begin{pmatrix} \gamma & -\beta\gamma & 0 & 0 \\ -\beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \gamma^2 - \beta^2\gamma^2 = 1$

$\Rightarrow d^4x' = d^4x \Rightarrow dx'^0 d^3x' = dx^0 d^3x \Rightarrow$

for $\rho\, d^3x = \rho' d^3x' \Rightarrow$ need ρ to transform like x^0

$\Rightarrow$ zero-component of a 4-vector.

$\Rightarrow$ <u>Current density</u> $\vec{J} = \rho \vec{v} = \rho \frac{d\vec{x}}{dt} \Rightarrow$ if

$\rho \sim x^0$, $t \sim x^0 \Rightarrow \vec{J}$ is a 3-vector

Back to Maxwell equations in Lorentz gauge:

as $\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \vec{\nabla}^2 = \partial_\mu \partial^\mu \Rightarrow$ it is a Lorentz-scalar and Maxwell equations become:

$$\begin{cases} \partial_\mu \partial^\mu \Phi = 4\pi \rho & \Rightarrow \Phi \sim \rho \sim x^0 \\ \partial_\mu \partial^\mu \vec{A} = \frac{4\pi}{c} \vec{J} & \Rightarrow \vec{A} \sim \vec{J} \sim \vec{x} \end{cases}$$

with Lorentz gauge condition $\partial_\mu A^\mu = 0$.

$\Rightarrow$ we can define a 4-vector potential: $\boxed{A^\mu = (\Phi, \vec{A})}$

Now Lorentz gauge condition is $\partial_\mu A^\mu = 0$

~ manifestly covariant, sometimes called covariant gauge. (cf. $\vec{\nabla}\cdot\vec{A} = 0$ Coulomb gauge)

Maxwell equations become $\partial_\mu \partial^\mu A^\nu = \frac{4\pi}{c} J^\nu$

or, defining $\Box \equiv \partial_\mu \partial^\mu$, $\boxed{\Box A^\mu = \frac{4\pi}{c} J^\mu}$.

Now let's express $\vec{E}$ and $\vec{B}$ in this covariant notation: $\vec{E} = -\frac{\partial \vec{A}}{\partial x^0} - \vec{\nabla}\Phi \Rightarrow$ say $E_x = -\frac{\partial \overbrace{A_x}^{=-A_1}}{\partial x^0} - \frac{\partial A_0}{\partial x^1} =$

$= +\partial_0 A_1 - \partial_1 A_0 \Rightarrow E^x = -\partial^0 A^1 + \partial^1 A^0 \Rightarrow$

$\Rightarrow E^i = -\left(\partial^0 A^i - \partial^i A^0\right)$

$B^i = \varepsilon^{ijk} \partial_j A^k \Rightarrow B^1 = \partial^2 A^3 - \partial^3 A^2$.

$\Rightarrow$ Define field-strength tensor

$$F^{\mu\nu} \equiv \partial^\mu A^\nu - \partial^\nu A^\mu$$

anti-symmetric ($F^{\mu\nu} = -F^{\nu\mu}$), 2nd rank tensor

$$\Rightarrow F^{\mu\nu} = \begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & -B_z & B_y \\ E_y & B_z & 0 & -B_x \\ E_z & -B_y & B_x & 0 \end{pmatrix}$$

$$F_{\mu\nu} = \begin{pmatrix} 0 & E_x & E_y & E_z \\ -E_x & 0 & -B_z & B_y \\ -E_y & B_z & 0 & -B_x \\ -E_z & -B_y & B_x & 0 \end{pmatrix}$$