Maxwell equations become $\partial_\mu \partial^\mu A^\nu = \frac{4\pi}{c} J^\nu$

or, defining $\Box \equiv \partial_\mu \partial^\mu$, $\boxed{\Box A^\mu = \frac{4\pi}{c} J^\mu}$.

Now let's express $\vec{E}$ and $\vec{B}$ in this covariant notation: $\vec{E} = -\frac{\partial \vec{A}}{\partial x^0} - \vec{\nabla}\Phi \Rightarrow$ say $E_x = -\frac{\partial \overbrace{A_x}^{=-A_1}}{\partial x^0} - \frac{\partial A_0}{\partial x^1} =$

$= +\partial_0 A_1 - \partial_1 A_0 \Rightarrow E^x = -\partial^0 A^1 + \partial^1 A^0 \Rightarrow$

$\Rightarrow E^i = -(\partial^0 A^i - \partial^i A^0)$

$B^i \overset{= \varepsilon^{ijk}\partial_j A^k}{=} -\varepsilon^{ijk} \partial^j A^k \Rightarrow B^1 = -(\partial^2 A^3 - \partial^3 A^2)$ $\left(\begin{array}{l} B_z = \partial_x A_y - \partial_y A_x = \\ = \partial_1 A^2 - \partial_2 A^1 \Rightarrow \\ \Rightarrow B^3 = -(\partial^1 A^2 - \partial^2 A^1) \end{array}\right)$

$\Rightarrow$ Define field-strength tensor

$$F^{\mu\nu} \equiv \partial^\mu A^\nu - \partial^\nu A^\mu$$

anti-symmetric ($F^{\mu\nu} = -F^{\nu\mu}$), 2nd rank tensor

$$\Rightarrow F^{\mu\nu} = \begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & -B_z & B_y \\ E_y & B_z & 0 & -B_x \\ E_z & -B_y & B_x & 0 \end{pmatrix}$$

$$F_{\mu\nu} = \begin{pmatrix} 0 & E_x & E_y & E_z \\ -E_x & 0 & -B_z & B_y \\ -E_y & B_z & 0 & -B_x \\ -E_z & -B_y & B_x & 0 \end{pmatrix}$$

$\Rightarrow$ Gauss's law & Ampere's law can be summarized by $\boxed{\partial_\mu F^{\mu\nu} = \frac{4\pi}{c} J^\nu}$

(in Lorenz gauge $\partial_\mu F^{\mu\nu} = \partial_\mu \partial^\mu A^\nu - \underbrace{\partial_\mu \partial^\nu A^\mu}_{=0 \text{ as } \partial_\mu A^\mu = 0} =$

$= \Box A^\nu = \frac{4\pi}{c} J^\nu$ ~ ok)

To write Faraday's law and $\vec{\nabla} \cdot \vec{B} = 0$ in a similar fashion, <u>define</u> a dual tensor

$$\boxed{\tilde{F}^{\mu\nu} = \frac{1}{2} \varepsilon^{\mu\nu\rho\sigma} F_{\rho\sigma}}$$

where $\varepsilon^{0123} = 1$, $\varepsilon^{\mu\nu\rho\sigma} = -\varepsilon^{\nu\mu\rho\sigma}$ ~ changes sign under permutations and $\varepsilon^{\mu\alpha\rho\alpha} = \varepsilon^{\alpha\alpha\rho\sigma} = \dots = 0$ (no 2 indeces are the same)

$$\tilde{F}^{\mu\nu} = \begin{pmatrix} 0 & -B_x & -B_y & -B_z \\ B_x & 0 & E_z & -E_y \\ B_y & -E_z & 0 & E_x \\ B_z & E_y & -E_x & 0 \end{pmatrix}$$

duality transform
$\vec{E} \to \vec{B}$
$\vec{B} \to -\vec{E}$.

$\Rightarrow$ the Faraday's law & $\vec{\nabla} \cdot \vec{B} = 0$ are ~~defined~~ written by $\boxed{\partial_\mu \tilde{F}^{\mu\nu} = 0}$

Thus the Maxwell equations now become

$$\boxed{\begin{aligned} \partial_\mu F^{\mu\nu} &= \frac{4\pi}{c} J^\nu \\ \partial_\mu \tilde{F}^{\mu\nu} &= 0 \end{aligned}}$$

(the second one is really needed to define $\vec{E}, \vec{B}$ in terms of A_μ $\Rightarrow$ only the 1st one is called Maxwell eqn's usually).

Transformation of $\vec{E}$ & $\vec{B}$ under Boosts.

$$F^{\mu'\nu'} = \Lambda^{\mu'}{}_{\mu} \Lambda^{\nu'}{}_{\nu} F^{\mu\nu} = \Lambda^{\mu'}{}_{\mu} F^{\mu\nu} \Lambda_{\nu}{}^{\nu'}$$

as $\Lambda^{\nu'}{}_{\nu} = \Lambda_\nu{}^{\nu'}$

$$\Rightarrow F^{\mu'\nu'} = \begin{pmatrix} \gamma & -\beta\gamma & 0 & 0 \\ -\beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & -B_z & B_y \\ E_y & B_z & 0 & -B_x \\ E_z & -B_y & B_x & 0 \end{pmatrix} \cdot$$

$$\begin{pmatrix} \gamma & -\beta\gamma & 0 & 0 \\ -\beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & (\beta^2\gamma^2 - \gamma^2)E_1 & -\gamma E_2 + \beta\gamma B_3 & -\gamma E_3 - \beta\gamma B_2 \\ (\gamma^2 - \beta^2\gamma^2)E_1 & 0 & \beta\gamma E_2 - \gamma B_3 & \beta\gamma E_3 + \gamma B_2 \\ \gamma E_2 - \beta\gamma B_3 & -\beta\gamma E_2 + \gamma B_3 & 0 & -B_1 \\ \gamma E_3 + \beta\gamma B_2 & -\beta\gamma E_3 - \gamma B_2 & B_1 & 0 \end{pmatrix}$$

$\Rightarrow$

$$\boxed{\begin{aligned} E_1' &= E_1 & B_1' &= B_1 \\ E_2' &= \gamma(E_2 - \beta B_3) & B_2' &= \gamma(B_2 + \beta E_3) \\ E_3' &= \gamma(E_3 + \beta B_2) & B_3' &= \gamma(B_3 - \beta E_2) \end{aligned}}$$

if $v \ll c \Rightarrow$ get $\vec{E}' = \vec{E} + \frac{v}{c} \times \vec{B}$ ~ cf. 1st quarter

$$\vec{B}' = \vec{B} - \frac{1}{c}\vec{v} \times \vec{E}.$$

Lorentz - invariants:

$F^{\mu\nu} F_{\mu\nu} = 2\left(\vec{B}^2 - \vec{E}^2\right)$ ~ by construction this is Lorentz-inv.

$F^{\mu\nu} \tilde{F}_{\mu\nu} = 4\, \vec{B} \cdot \vec{E}$ ~ also Lorentz-inv.

Example: plane waves, $\vec{E} = \frac{c}{\omega} \vec{k} \times \vec{B} \Rightarrow$

$\Rightarrow \vec{E} \cdot \vec{B} = 0$, $|\vec{E}| = |\vec{B}| \Rightarrow E^2 - B^2 = 0$

~ true in all frames!

Example: moving point charge: in it's rest frame the field is given by Coulomb's law: $\vec{E} = \frac{q}{r^3}\vec{r}$

(note Gaussian units)