$$\Rightarrow \vec{B} = \frac{q}{c} \frac{\vec{v}\times\vec{r}}{r^3} \sim \text{Biot-Savart law!}$$

In the UR limit, $\beta \to 1$, $\gamma \to \infty$ $\Rightarrow$

$$\Rightarrow B_y \approx -\frac{q\,z}{\gamma^2 (x-vt)^3} = -E_z \; ; \quad B_z \approx \frac{q\,y}{(x-vt)^3} = E_y \; ;$$

$$E_x \approx \frac{q}{\gamma^2 (x-vt)^2}\, \mathrm{Sign}(x-vt) \,.$$

Relativistic Particles in Electromagnetic Fields.

Start with Lorentz force:

$$\frac{d\vec{p}}{dt} = q\left(\vec{E} + \frac{1}{c}\vec{v}\times\vec{B}\right) = \vec{F}$$

$\Rightarrow$ want to transform this into an equation for 3-components of 4-vector p^μ $\Rightarrow$

$$\Rightarrow p^\mu = m u^\mu = m\gamma(c, \vec{v}) \Rightarrow$$

$$\frac{d\vec{p}}{d\tau} = \gamma \frac{d\vec{p}}{dt} = \frac{q}{c}\left[u^0 \vec{E} + \vec{u}\times\vec{B}\right]$$

On the other hand $\dfrac{dE}{dt} = \vec{F}\cdot\vec{v}$ $\Rightarrow$

$$\Rightarrow \frac{dp^0}{d\tau} = \gamma \frac{1}{c} \frac{dE}{dt} = \frac{\gamma}{c} q \vec{v} \cdot \left(\vec{E} + \frac{1}{c} \vec{v} \times \vec{B} \right) =$$

$$= \frac{q}{c} \gamma \vec{v} \cdot \vec{E} \Rightarrow \text{as } E^i = -F^{0i} \Rightarrow$$

$$\Rightarrow \gamma \vec{v} \cdot \vec{E} = -\gamma v_i E^i = \gamma v_i F^{0i} = u_i F^{0i} =$$

$$= u_\mu F^{0\mu} \text{ as } F^{00} = 0 \Rightarrow \frac{dp^0}{d\tau} = \frac{q}{c} u_\mu F^{0\mu}$$

Also, $\dfrac{dp^i}{d\tau} = \dfrac{q}{c} u_\mu F^{i\mu}$ (check)

$$\Rightarrow \boxed{\frac{dp^\mu}{d\tau} = \frac{q}{c} u_\nu F^{\mu\nu}}$$

Lorentz-covariant formulation of Lorentz force.

$\Rightarrow$ What does this mean for the Lagrangian?

Remember, $L_{free} = -mc^2 \sqrt{1 - \frac{v^2}{c^2}}$.

If an action is given by $S = \int_{t_1}^{t_2} dt\, L[q_i(t), \dot{q}_i(t), t]$

$\Rightarrow$ least action principle gives

$$\boxed{\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} = 0.}$$

Apply this to L_{free}: $q_i(t) = x^i(t)$,

$\dot{q}_i(t) = \frac{d}{dt} x^i(t) = \dot{x}^i(t) = v^i(t) \Rightarrow L_{free} = L(\dot{x}^i)$

$\Rightarrow$ get $\frac{d}{dt}\left(m\gamma\vec{v}\right) = 0 \Rightarrow \frac{d\vec{p}}{dt} = 0. \Rightarrow m\gamma\vec{v} = \text{const.}$

Let's try to construct the interaction Lagrangian between point charges & E&M fields.

In the NR limit $L = T - V \Rightarrow$ the potential energy of a point charge in electric field is $V = e\Phi \Rightarrow L \simeq -e\Phi$

$\Rightarrow$ however, $\Phi = A^0$ and ~~~~ we need to have a covariant expression $\Rightarrow \Phi \to A^\mu \Rightarrow$

$\Rightarrow$ need to multiply by a 4-vector $\Rightarrow$

$\Rightarrow$ only have u^μ (can't have x^μ ~ would have "prefferred" coordinates $\Rightarrow$ not physically meaningfull) $\Rightarrow$ get $L \simeq -\frac{e}{c} u_\mu A^\mu$.

Still, the action $S = \int dt \cdot L$ is Lorentz-scalar

$\Rightarrow S = \int d\tau \cdot \gamma \cdot L$ is salar $\Rightarrow \gamma L$ is scalar $\Rightarrow$

$$\Rightarrow \quad L_{int} = -\frac{e}{c\gamma} u_\mu A^\mu$$

$$\Rightarrow \quad S_{int} = -\frac{e}{c}\int_{t_1}^{t_2} dt \, \frac{1}{\gamma} u_\mu A^\mu = -\frac{e}{c}\int_1^2 dx_\mu A^\mu$$

The total Lagrangian is

$$L = -mc^2\sqrt{1-\frac{v^2}{c^2}} - \frac{e}{c\gamma} u_\mu A^\mu \Rightarrow$$

$$\boxed{L = -mc^2\sqrt{1-\frac{v^2}{c^2}} - e\Phi + \frac{e}{c}\vec{v}\cdot\vec{A}}$$

Equations of motion:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial v^i}\right) - \frac{\partial L}{\partial x^i} = 0 \Rightarrow \frac{d}{dt}\left(\frac{mv^i}{\sqrt{1-\frac{v^2}{c^2}}}\right) + e\frac{\partial \Phi}{\partial x^i} -$$

$$-\frac{e}{c}\vec{v}\cdot\frac{\partial \vec{A}}{\partial x^i} + \frac{e}{c}\frac{dA^i}{dt} = 0 \Rightarrow \text{as } \frac{dA^i}{dt} = \frac{\partial A^i}{\partial t} + \frac{\partial x_j}{\partial t}\frac{\partial A^i}{\partial x_j}$$

$$\frac{dp^i}{dt} = -\frac{e}{c}\Big[c\frac{\partial A^0}{\partial x^i} - v^j\frac{\partial A^j}{\partial x^i} + c\frac{\partial A^i}{\partial x^0} +$$

$$+ v_j\frac{\partial A^i}{\partial x_j}\Big] = -\frac{e}{c}\Big[c\overbrace{\left(-\frac{\partial A^0}{\partial x_i} + \frac{\partial A^i}{\partial x^0}\right)}^{F^{0i}} - v_j\overbrace{\left(\frac{\partial A^j}{\partial x_i} - \frac{\partial A^i}{\partial x_j}\right)}^{F^{ij}}\Big]$$

$$= \frac{e}{\gamma c} u_\mu F^{i\mu} \Rightarrow \frac{dp^i}{d\tau} = \frac{e}{c} u_\mu F^{i\mu}, \text{ as desired!}$$