Last time | Functional integral quantization (φ^4)

Free theory:

$$Z_0[j] = \int \mathcal{D}\varphi \; e^{i\int d^4x \left[\frac{1}{2}\partial_\mu\varphi\partial^\mu\varphi - \frac{m^2}{2}\varphi^2 + j\varphi\right]}$$

$$\Rightarrow \quad Z_0[j] = \frac{1}{\sqrt{\det(i\hat{D})}} \, e^{-\frac{1}{2}\int d^4x\, d^4y\, j(x) D_F(x-y) j(y)}$$

where $\hat{D} = \Box + m^2 - i\varepsilon$, $\quad D_F(x-y) = \int \frac{d^4p}{(2\pi)^4} e^{-ip\cdot(x-y)} \frac{i}{p^2 - m^2 + i\varepsilon}$

Feynman propagator

Interacting theory:

$$Z[j] = \int \mathcal{D}\varphi \, e^{i\int d^4x \left[\frac{1}{2}\partial_\mu\varphi\partial^\mu\varphi - \frac{m^2}{2}\varphi^2 - \frac{\lambda}{4!}\varphi^4 + j\varphi\right]}$$

$$= e^{-i\frac{\lambda}{4!}\int d^4x \frac{\delta^4}{\delta j(x)^4}} \; Z_0[j]$$

$\Rightarrow$ expand in λ $\Rightarrow$ get Feynman diagrams

$$\langle\Psi_0| T\varphi(x_1)\varphi(x_2)|\Psi_0\rangle = \frac{1}{Z[0]}(-i)^2 \frac{\delta^2}{\delta j(x_1)\,\delta j(x_2)} Z[j]\Big|_{j=0}$$

↑

this is how you find expectation values.

Faddeev - Popov Quantization

We want to quantize a gauge theory:

$$\mathcal{L} = -\frac{1}{4} F^a_{\mu\nu} F^{a\mu\nu}$$

(consider a general non-Abelian case).

The generating functional is

$$Z[0] = \int \mathcal{D} A_\mu \, e^{iS} = \int \mathcal{D} A_\mu \, e^{i\int d^4x \left(-\frac{1}{4}\right) F^a_{\mu\nu} F^{a\mu\nu}} =$$

$$= \int \mathcal{D} \bar{A}_\mu \, e^{iS} \cdot \int \mathcal{D} \Lambda$$

where $\bar{A}_\mu$ is the field in one particular gauge, Λ is the gauge transformation.

Problem: $\int \mathcal{D}\Lambda = \infty \Rightarrow Z = \infty \Rightarrow$ bad!

Even worse is the need to pick a gauge: consider abelian field A_μ: $\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} = -\frac{1}{4}(\partial_\mu A_\nu - \partial_\nu A_\mu)(\partial^\mu A^\nu - \partial^\nu A^\mu) = \frac{1}{2} A^\mu \underbrace{\left[g_{\mu\nu} \partial^2 - \partial_\mu \partial_\nu \right]}_{(D^{-1})_{\mu\nu}} A^\nu$

$\Rightarrow$ to find photon propagator need to solve

$$\left[g_{\mu\nu} \partial^2 - \partial_\mu \partial_\nu \right] D^{\nu\rho}(x) = \delta_\mu^\rho \, \delta^4(x)$$

$\Rightarrow$ act with ∂^μ $\Rightarrow$ $(\partial_\nu \partial^2 - \partial^2 \partial_\nu) D^{\nu\rho} = 0 = \partial^\rho \delta^4(x)$

$\Rightarrow$ this can not be true $\Rightarrow$ the operator has no inverse! $\Rightarrow$ no photon propagator?

However, if we choose a gauge, e.g. $\partial_\mu A_\mu = 0$

$$\Rightarrow \quad \mathcal{L} = \frac{1}{2} A^\mu \Box A^\nu \Rightarrow g_{\mu\nu} \Box D^{\nu\rho}(x) = \delta_\mu{}^\rho \delta^4(x).$$

$\Rightarrow$ easy to invert!

$\Rightarrow$ Need to fix the gauge!

Start with $\quad Z(0) = \int \mathcal{D} A_\mu \, e^{iS'}$.

Insert into the integrand $\left(A_\mu{}^\Lambda = \Lambda A_\mu \Lambda^\dagger - \frac{i}{g}(\partial_\mu \Lambda)\Lambda^\dagger\right)$

$$1 = \int \mathcal{D}\Lambda \; \delta(\Lambda - 1) = \int \mathcal{D}\Lambda \cdot \delta(G(A^\Lambda)) \det\left(\frac{\delta G(A^\Lambda)}{\delta \Lambda}\right)$$

eval. at $\Lambda = 1$ due to δ-fct

where $G(A) = 0$ is the gauge condition we want to impose, e.g. $G(A) = \partial_\mu A^\mu$ for covariant gauge. Now

$$Z(0) = \int \mathcal{D} A_\mu \, e^{iS'(A_\mu)} \int \mathcal{D}\Lambda \; \delta(G(A^\Lambda)) \det\left(\frac{\delta G(A^\Lambda)}{\delta \Lambda}\right)\Bigg|_{\Lambda = 1}$$

Change the order of integration & define a new field $A_\mu' = A_\mu^\Lambda$ to write (dropping the prime)

(as in QED $\mathcal{D} A_\mu = \mathcal{D} A_\mu^\Lambda$, $S'(A_\mu) = S'(A_\mu^\Lambda)$) Λ = unitary

$$Z(0) = \underbrace{\int \mathcal{D}\Lambda}_{\text{still } \infty \text{, but}} \cdot \int \mathcal{D}A_\mu \, e^{iS(A_\mu)} \, \delta(G(A_\mu)) \underbrace{\det\left(\frac{\delta G(A^\Lambda)}{\delta \Lambda}\right)}_{\text{we assumed that this is independent of } \Lambda \text{, only a ftn of } A^\Lambda \to A} \quad (24)$$

an overall factor $\Rightarrow$ cancels in correlators like

$$\langle A_\mu(x) A_\nu(y) \rangle = \frac{1}{Z} \cdot \int \mathcal{D}A_\mu \, A_\mu(x) A_\nu(y) \, e^{iS}.$$

a trick: choose $G(A) = \partial_\mu A^{a\mu} - \omega^a(x) \Rightarrow$

$$\Rightarrow \delta(G(A)) = \delta(\overbrace{\partial_\mu A^{a\mu}}^{\bar{G}(A)} - \omega^a(x))$$

Nothing else in Z depends on $\omega^a(x)$ $\Rightarrow$ integrate over $\omega(x)$:

$$1 = \underbrace{N(\xi)}_{\text{norm}} \int \mathcal{D}\omega^a(x) \, e^{-i\int d^4x \frac{\omega^{a2}}{2\xi}}$$

$$\Rightarrow Z(0) = \int \mathcal{D}\Lambda \cdot N(\xi) \int \mathcal{D}A_\mu \, e^{iS(A_\mu)} \int \mathcal{D}\omega^a \, e^{-i\int d^4x \frac{\omega^{a2}}{2\xi}}.$$

$$\cdot \, \delta(\bar{G}(A) - \omega(x)) \det\left(\frac{\delta G(A^\Lambda)}{\delta \Lambda}\right) = \int \mathcal{D}\Lambda \cdot N(\xi) \cdot$$

drop $\Lambda = 1$ as δ-ftn insures it

$$\cdot \int \mathcal{D}A_\mu \cdot \det\left(\frac{\delta G(A^\Lambda)}{\delta \Lambda}\right) \cdot e^{iS(A_\mu)} \cdot e^{-i\int d^4x \frac{1}{2\xi}(\bar{G}(A))^2}.$$

$\int \mathcal{D}\Lambda \cdot N(\xi)$ is an unimportant overall factor.

What do we do with $\det\left(\frac{\delta G}{\delta \Lambda}\right)$?

We have

$$Z(0) \sim \int \mathcal{D} A_\mu \det\left(\frac{\delta G(A^\Lambda)}{\delta \Lambda}\right) \cdot e^{i S(A) - i \int d^4x \frac{[G(A)]^2}{2\xi}}$$

We want to put det into the exponent ~ make it a part of the Lagrangian.

Note that $\int_{-\infty}^{\infty} dx \, e^{-a x^2} = \sqrt{\frac{\pi}{a}}$ =>

$$\Rightarrow \int_{-\infty}^{\infty} dx_1 \ldots dx_n \, e^{-a_1 x_1^2 - \ldots - a_n x_n^2} = \left(\pi\right)^{n/2} \cdot \frac{1}{\sqrt{a_1 a_2 \ldots a_n}}$$

$$= \frac{\pi^{n/2}}{\sqrt{\det A}}, \quad A = \begin{pmatrix} a_1 & & 0 \\ & a_2 & \\ 0 & & \ddots \; a_n \end{pmatrix}$$ a diagonal matrix.

Similarly, for $\forall A$ get $\boxed{\int_{-\infty}^{\infty} d^n x \, e^{-x^T A x} = \frac{\pi^{n/2}}{\sqrt{\det A}}}$

=> can absorb $\frac{1}{\sqrt{\det A}}$ into exponent. But here we have $\det A$!

Grassmann #'s: $\eta \cdot \theta = -\theta \cdot \eta$ (anti-commute)
$\Rightarrow \theta^2 = 0, \eta^2 = 0$

Grassmann quantities: η is a Grassmann #

A, B = regular or Grassmann #'s

=> if η is single-component => $f(\eta) = A + B\eta$

$(\eta^2 = 0, \; \eta^3 = 0, \ldots)$ => $\frac{df}{d\eta} = B$ => $\frac{d^2 f}{d\eta^2} = 0$

if B = complex (regular) #