Last time | **Faddeev-Popov Quantization (cont'd)**

$$Z[0] \overset{?}{=} \int \mathcal{D}A_\mu \, e^{iS} \Rightarrow$$ problems, no propagator until one fixes the gauge

$$Z[0] = \int \mathcal{D}A_\mu \, e^{iS(A_\mu)} \int \mathcal{D}\Lambda \, \delta(G(A^\Lambda)) \, \det\left(\frac{\delta G(A^\Lambda)}{\delta \Lambda}\right)\Bigg|_{\Lambda=1}$$

$$A_\mu^\Lambda = \Lambda A_\mu \Lambda^{-1} - \frac{i}{g}(\partial_\mu \Lambda)\Lambda^{-1}$$

$\Rightarrow$ need to absorb δ-function & det into exponent.

$\Rightarrow$ We arrived at

$$Z[0] = \int \mathcal{D}\Lambda \, N(\xi) \int \mathcal{D}A_\mu \, \det\left(\frac{\delta G(A^\Lambda)}{\delta\Lambda}\right)\Bigg|_{\Lambda=1} e^{iS(A_\mu) - i\int d^4x \frac{1}{2\xi}(\partial_\mu A^{a\mu})^2}$$

$\nwarrow$ can drop, δ-ftn insured this

for $\partial_\mu A^{a\mu} = \omega^a(x)$ types of gauges

$$Z(0) = \int \mathcal{D}\Lambda \cdot \int \mathcal{D} A_\mu \, e^{i S(A_\mu)} \, \delta(G(A_\mu)) \underbrace{\det\left(\frac{\delta G(A^\Lambda)}{\delta \Lambda}\right)}_{\text{we assumed that this is independent of } \Lambda \text{, only after } A^\Lambda \to A} \quad (24)$$

$\underbrace{\int \mathcal{D}\Lambda}$ still ∞, but an overall factor $\Rightarrow$ cancels in correlators like

$$\langle A_\mu(x) A_\nu(y) \rangle = \frac{1}{Z} \cdot \int \mathcal{D} A_\mu \, A_\mu(x) A_\nu(y) \, e^{iS'}.$$

a trick: choose $G(A) = \partial_\mu A^{a\mu} - \omega^a(x) \Rightarrow$

$$\Rightarrow \delta(G(A)) = \delta(\overbrace{\partial_\mu A^{a\mu}}^{\bar{G}(A)} - \omega^a(x))$$

Nothing else in Z depends on $\omega^a(x)$ $\Rightarrow$ integrate over $\omega(x)$:

$$1 = \underbrace{N(\xi)}_{\text{Norm}} \int \mathcal{D}\omega^a(x) \, e^{-i\int d^4x \frac{\omega^{a2}}{2\xi}}$$

$$\Rightarrow Z(0) = \int \mathcal{D}\Lambda \cdot N(\xi) \int \mathcal{D} A_\mu \, e^{iS'(A_\mu)} \int \mathcal{D}\omega^a \, e^{-i\int d^4x \frac{\omega^{a2}}{2\xi}}$$

$$\cdot \, \delta(\bar{G}(A) - \omega(x)) \det\left(\frac{\delta G(A^\Lambda)}{\delta\Lambda}\right) = \int \mathcal{D}\Lambda \cdot N(\xi) \cdot$$

drop $A^\Lambda = A$ as δ-ftn insures it

$$\cdot \int \mathcal{D} A_\mu \cdot \det\left(\frac{\delta G(A^\Lambda)}{\delta\Lambda}\right) \cdot e^{iS'(A_\mu)} \cdot e^{-i\int d^4x \frac{1}{2\xi}(\bar{G}(A))^2}.$$

$\int \mathcal{D}\Lambda \cdot N(\xi)$ is an unimportant overall factor.

What do we do with $\det\left(\frac{\delta G}{\delta \Lambda}\right)$?

We have

$$Z(0) \sim \int \mathcal{D}A_\mu \det\left(\frac{\delta G(A^\Lambda)}{\delta \Lambda}\right) \cdot e^{i S(A) - i \int d^4x \frac{[G(A)]^2}{2\xi}}$$

We want to put det into the exponent ~ make it a part of the Lagrangian.

Note that $\int_{-\infty}^{\infty} dx\, e^{-ax^2} = \sqrt{\frac{\pi}{a}} \Rightarrow$

$$\Rightarrow \int_{-\infty}^{\infty} dx_1 \ldots dx_n\, e^{-a_1 x_1^2 - \ldots - a_n x_n^2} = \left(\pi\right)^{n/2} \cdot \frac{1}{\sqrt{a_1 a_2 \ldots a_n}}$$

$$= \frac{\pi^{n/2}}{\sqrt{\det A}}, \quad A = \begin{pmatrix} a_1 & & 0 \\ & a_2 & \\ 0 & & \ddots\, a_n \end{pmatrix} \text{ a diagonal matrix.}$$

Similarly, for $\forall A$ get $\boxed{\int_{-\infty}^{\infty} d^n x\, e^{-x^T A x} = \frac{\pi^{n/2}}{\sqrt{\det A}}}$

$\Rightarrow$ can absorb $\frac{1}{\sqrt{\det A}}$ into exponent. But here we have $\det A$!

Grassmann #'s: $\eta \cdot \theta = -\theta \cdot \eta$ (anti-commute)
$\Rightarrow \theta^2 = 0,\ \eta^2 = 0$

Grassmann quantities: η is a Grassmann #

A, B = regular or Grassmann #'s

$\Rightarrow$ if η is single-component $\Rightarrow f(\eta) = A + B\eta$

$(\eta^2 = 0,\ \eta^3 = 0, \ldots) \Rightarrow \frac{df}{d\eta} = B \Rightarrow \frac{d^2 f}{d\eta^2} = 0$

if B = complex (regular) #

$\Rightarrow$ no inverse to differentiation?

To define integrals note:

$$\int d\eta\, f(\eta) = \int d\eta\, (A + B\eta) = \Big|\, \eta \to \eta + \theta = \int d\eta\, (A + B\eta + B\theta)$$

$$\Rightarrow \int d\eta \cdot A = \int d\eta\, (A + B\theta) \Rightarrow \int d\eta = 0$$

$$\int d\eta\, B\eta = B \quad \text{(linear in } B\text{, adjust constant to 1)}$$

$$\Rightarrow \int d\eta \cdot \eta = 1.$$ Note that $\int d\theta \int d\eta \cdot \eta \cdot \theta = +1$.

Complex $\eta = \frac{\eta_1 + i\eta_2}{\sqrt{2}}$: $\bar{\eta} = \frac{\eta_1 - i\eta_2}{\sqrt{2}}$, $\eta_1, \eta_2 =$ Grassmann #'s, $\eta^2 = 0$, $\bar{\eta}^2 = 0$, $\bar{\eta}\eta \neq 0$; $\bar{\eta}$ is c.c. $\int d\eta = \int d\bar{\eta} = 0$, $\int d\eta \cdot \eta = \int d\bar{\eta}\, \bar{\eta} = 1$.

$$\int d\bar{\eta}\, d\eta\, e^{-b\bar{\eta}\eta} = \int d\bar{\eta} \int d\eta\, (1 - b\bar{\eta}\eta) = \int d\bar{\eta} \cdot (+) b\bar{\eta} = 1\, b = b.$$

Two-component Grassmann #'s: $\eta = \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix}$, $\bar{\eta} = \begin{pmatrix} \bar{\eta}_1 \\ \bar{\eta}_2 \end{pmatrix}$

$$\Rightarrow \bar{\eta}^T \eta = \bar{\eta}_1 \eta_1 + \bar{\eta}_2 \eta_2$$

$$(\bar{\eta}^T \eta)^2 = (\bar{\eta}_1 \eta_1 + \bar{\eta}_2 \eta_2)^2 = 2\bar{\eta}_1 \eta_1 \bar{\eta}_2 \eta_2$$

$\int d\bar{\eta}\, d\eta\, e^{-\bar{\eta}^T M \eta}$ need to find, M is a 2×2 matrix

$$\Rightarrow e^{-\bar{\eta}^T M \eta} = e^{-(\bar{\eta}_1\ \bar{\eta}_2)\begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix}\begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix}} =$$

$$= \exp\left\{ -(\bar{\eta}_1\ \bar{\eta}_2) \begin{pmatrix} M_{11}\eta_1 + M_{12}\eta_2 \\ M_{21}\eta_1 + M_{22}\eta_2 \end{pmatrix} \right\} = \exp\Big\{ -\big[\bar{\eta}_1 (M_{11}\eta_1 + M_{12}\eta_2) +$$

$$+ \bar{\eta}_2 \left(M_{21} \eta_1 + M_{22} \eta_2 \right) \Big] \Big\} = 1 - \bar{\eta}_1 (M_{11}\eta_1 + M_{12}\eta_2)$$

$$- \bar{\eta}_2 (M_{21}\eta_1 + M_{22}\eta_2) + \frac{1}{2} 2 \bar{\eta}_1 (M_{11}\eta_1 + M_{12}\eta_2) \bar{\eta}_2 (M_{21}\eta_1$$

$$+ M_{22}\eta_2) = 1 - \bar{\eta}_1 (M_{11}\eta_1 + M_{12}\eta_2) - \bar{\eta}_2 (M_{21}\eta_1 + M_{22}\eta_2)$$

$$+ \bar{\eta}_1 \eta_1 \bar{\eta}_2 \eta_2 \left(M_{11} M_{22} - M_{12} M_{21} \right)$$

$$\Rightarrow \int d\bar{\eta}\, d\eta\, e^{-\bar{\eta}^T M \eta} = \int \overbrace{d\bar{\eta}_1\, d\bar{\eta}_2\, d\eta_2\, d\eta_1}^{d\bar{\eta}_1 d\eta_1\, d\bar{\eta}_2 d\eta_2} \bar{\eta}_1 \eta_1 \bar{\eta}_2 \eta_2 \cdot$$

$$\cdot (M_{11} M_{22} - M_{12} M_{21}) = \det M.$$

$\Rightarrow$ can show for $\forall$ dimension

$$\int d\bar{\eta}\, d\eta\, e^{-\bar{\eta}^T M \eta} = \det M$$

$$\Rightarrow \det \left[\frac{\delta G(A^\Lambda)}{\delta \Lambda} \right] = \int \mathcal{D}\bar{\eta}\, \mathcal{D}\eta\, e^{-i \int d^4x\, \bar{\eta}^a \frac{\delta G}{\delta \Lambda} \eta^a}$$

$$\Rightarrow Z(0) \sim \int \mathcal{D} A_\mu \mathcal{D}\eta \mathcal{D}\bar{\eta}\, e^{i S(A) - i \int d^4x \frac{[\bar{G}(A)]^2}{2\xi} - i \int d^4x\, \bar{\eta} \frac{\delta G}{\delta \Lambda} \eta}$$

η^a ~ Faddeev-Popov ghost, has a color index $a = 1, \dots, N^2-1$ for $SU(N)$

Covariant gauge: $\bar{G}(A) = \partial_\mu A^{a\mu}$ $\Rightarrow$ gauge transform is $A_\mu \to \Lambda A_\mu \Lambda^{-1} - \frac{i}{g} (\partial_\mu \Lambda) \Lambda^{-1}$

$\Rightarrow$ for infinitesimal gauge transform:
(need $\frac{\delta G(A^\Lambda)}{\delta\Lambda} \Rightarrow$ Vary $\Lambda \to \Lambda + \delta\Lambda$, below A_μ is A_μ^Λ)

$$\Lambda = 1 + i\,\alpha^a T^a \Rightarrow A_\mu^a T^a \to (1 + i\,\alpha^a T^a) A_\mu \cdot$$

$$\cdot (1 - i\,\alpha^b T^b) - \frac{i}{g}\, i T^a (\partial_\mu \alpha^a)(1 - i\,\alpha^b T^b) =$$

$$= A_\mu + i[\alpha, A_\mu] + \frac{1}{g}\partial_\mu \alpha = T^a A_\mu'^a$$

$$\Rightarrow A_\mu'^a = A_\mu^a + i \cdot i f^{abc} \alpha^b A_\mu^c + \frac{1}{g}\partial_\mu \alpha^a$$

$$= A_\mu^a + f^{abc} A_\mu^b \alpha^c + \frac{1}{g}\partial_\mu \alpha^a =$$

$$= A_\mu^a + \frac{1}{g}\mathcal{D}_\mu \alpha^a, \text{ where } \mathcal{D}_\mu \alpha^a = \partial_\mu \alpha^a + g f^{abc} A_\mu^b \alpha^c$$

$$\Rightarrow \frac{\delta G}{\delta \Lambda} = \frac{\delta G}{\delta \alpha} = \frac{\delta(\partial_\mu A^{a\mu})}{\delta \alpha} = \partial_\mu \frac{1}{g}\mathcal{D}^\mu$$

absorb by re-defining η

$$\Rightarrow \mathcal{L} = -\frac{1}{4}F_{\mu\nu}^a F^{a\mu\nu} - \frac{1}{2\xi}(\partial_\mu A^{a\mu})^2 - \bar{\eta}\,\partial_\mu \mathcal{D}^\mu \eta$$

$$\Rightarrow \mathcal{L} = -\frac{1}{4}F_{\mu\nu}^a F^{a\mu\nu} - \frac{1}{2\xi}(\partial_\mu A^{a\mu})^2 + (\partial_\mu \bar{\eta}^a)(\mathcal{D}^\mu \eta)^a$$

Faddeev-Popov Lagrangian

<u>Light-cone gauge:</u> $\bar{G}(A) = n \cdot A^a = n_\mu A^{a\mu} \Rightarrow$

$$\Rightarrow \frac{\delta G}{\delta \alpha} = n_\mu \frac{\delta A^{a\mu}}{\delta \alpha} = n_\mu \frac{1}{g}\mathcal{D}^\mu = \frac{1}{g} n_\mu (\partial^\mu - ig[A^\mu, \ldots])$$

(in the $\xi \to 0$ limit) $\longrightarrow$ as $n \cdot A = 0$

$= \frac{1}{g} n \cdot \partial$ => is A_μ - independent =>

=> ~~again~~ gives only an overall factor in

=> do not need it, no need to introduce the ghost!

=> no ghosts in light-cone gauge!

$$\Rightarrow \mathcal{L} = -\frac{1}{4} F^a_{\mu\nu} F^{a\mu\nu} - \frac{1}{2\xi} (n \cdot A^a)^2$$ LC gauge Lagrangian.

($\xi \to 0$ is implicit)

=> unlike QED case, in QCD in covariant gauge $\frac{\delta G(A^\Lambda)}{\delta \Lambda}$ depends on A_μ and can not be taken out of DA_μ integral => have to introduce the ghost field!