Last time | Finished quantizing non-Abelian gauge theories. We obtained the generating functional for pure-glue theory:

$$\boxed{Z[j_\mu^a] = \int \mathcal{D}A_\mu \, \mathcal{D}\eta \, \mathcal{D}\bar{\eta} \; e^{i\int d^4x \left[\mathcal{L}_{FP} + j_\mu^a A^{a\mu}\right]}}$$

where the Faddeev - Popov Lagrangian is

$$\boxed{\mathcal{L}_{FP} = -\frac{1}{4} F_{\mu\nu}^a F^{a\mu\nu} - \frac{1}{2\xi}\left(\partial_\mu A^{a\mu}\right)^2 + \left(\partial_\mu \bar{\eta}^a\right)\left(\mathcal{D}^\mu \eta\right)^a}$$

(Lorenz gauge, $\partial_\mu A^\mu = 0$)

$$\mathcal{D}_\mu \alpha^a \equiv \partial_\mu \alpha^a + g f^{abc} A_\mu^b \alpha^c$$

η^a = ghost field, $a = 1, \dots, N^2-1$, Grassmann number in the integral.

Light-cone gauge: no ghost

$n_\mu A^\mu = 0$

$$\Rightarrow \boxed{\mathcal{L} = -\frac{1}{4} F_{\mu\nu}^a F^{a\mu\nu} - \frac{1}{2\xi}\left(n_\mu A^{a\mu}\right)^2}$$

$$\boxed{Z[j_\mu^a] = \lim_{\xi\to 0} \int \mathcal{D}A_\mu \; e^{i\int d^4x \left[\mathcal{L} + j_\mu^a A^{a\mu}\right]}}$$

Feynman Rules in QCD

$$\boxed{\mathcal{L}_{QCD} = \bar{q}^f [i\gamma^\mu D_\mu - m_f] q^f - \frac{1}{4} F^a_{\mu\nu} F^{a\mu\nu}}$$

However, this Lagrangian is gauge-invariant

$$\begin{cases} A_\mu \to S A_\mu S^{-1} - \frac{i}{g}(\partial_\mu S) S^{-1} \\ q \to S q \end{cases}$$

$\Rightarrow$ need to fix the gauge!

(i) Covariant (Lorenz) gauge $\partial_\mu A^{a\mu} = 0$

$\Rightarrow$ to fix the gauge need to introduce the so-called ghost fields:

$$\mathcal{L}_{QCD}^{cov.\,gauge} = \bar{q}^f [i\gamma^\mu D_\mu - m_f] q^f - \frac{1}{4} F^a_{\mu\nu} F^{a\mu\nu} - \frac{1}{2\xi}(\partial_\mu A^{a\mu})(\partial_\nu A^{a\nu}) + \partial_\mu \bar{\eta}^a \mathcal{D}^\mu \eta^a$$

η^a is a scalar field ~ Faddeev-Popov ghost

η^a is an anti-commuting field (Grassmann variables) (quantized like a fermion) $\Rightarrow$ unphysical $\Rightarrow$ ghosts

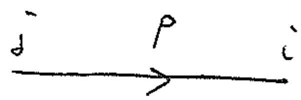
$\bar{\eta}^a$ is c.c. of η ; $\boxed{D_\mu = \partial_\mu - i g A_\mu}$

$$\eta = \sum_{a=1}^{8} T^a \eta^a \,, \qquad D_\mu \eta = \partial_\mu \eta - \underbrace{ig\,[A_\mu, \eta]}_{\text{note the commutator!}}$$

$$D_\mu \eta^a = \partial_\mu \eta^a + g\, f^{abc} A_\mu^b \eta^c$$

Feynman Rules:

Quark Propagator:

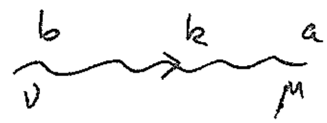


$$\frac{i}{\gamma \cdot p - m}\, \delta_{ij} = \frac{i(\gamma \cdot p + m)}{p^2 - m^2 + i\varepsilon}\, \delta_{ij}$$

Ghost Propagator:
$\bar{\eta}\eta$

$$\frac{i}{k^2 + i\varepsilon}\, \delta_{ab}$$

Gluon Propagator:

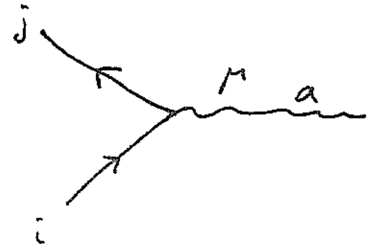


$$\frac{-i}{k^2 + i\varepsilon}\, \delta^{ab} \left[g_{\mu\nu} - (1-\xi) \frac{k_\mu k_\nu}{k^2} \right]$$

$\xi = 0$ Landau gauge

$\xi = 1$ Feynman gauge

Quark-Gluon Vertex:



$$ig\, \gamma^\mu \, (T^a)_{ji}$$

Other interaction vertices are less trivial:

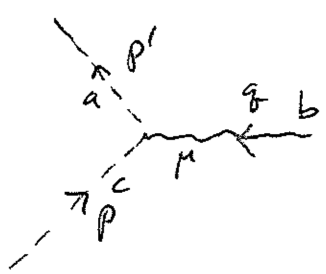
$$\partial_\mu \bar\eta D^\mu \eta = \partial_\mu \bar\eta \partial^\mu \eta \underbrace{-ig\, \partial_\mu \bar\eta [A^\mu, \eta]}_{\text{ghost-gluon interaction}}$$

$$\Rightarrow \mathcal{L}_{gG} = -ig\, \partial_\mu \bar\eta [A^\mu, \eta] = \cancel{-i}g\, \partial_\mu \bar\eta^a \cancel{i} f^{abc} A^{b\mu} \eta^c$$

$$= g (\partial_\mu \bar\eta^a) f^{abc} A^{b\mu} \eta^c$$

when contracting with $\eta(y)$: $\eta(y)\; g f^{abc} (\partial_\mu \bar\eta^a_{(x)}) A^{b\mu} \eta^c$, where the contraction of $\eta(y)$ with $\bar\eta^a(x)$ $= e^{-ip'\cdot(y-x)}$, and $\partial_\mu \to ip'_\mu$

$$\Rightarrow i g f^{abc} p'_\mu \otimes \overset{\text{from } iS}{i} \Rightarrow -g f^{abc} p'_\mu = g f^{bac} p'_\mu$$

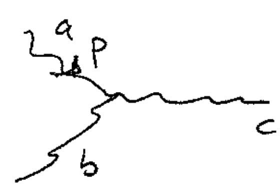


is the contribution of ghost-gluon vertex

3-gluon vertex: $-\frac{1}{4} F^a_{\mu\nu} F^{a\mu\nu} = -\frac{1}{2} (\partial_\mu A^a_\nu - \partial_\nu A^a_\mu) \cdot g \cdot f^{abc} A^b_\mu A^c_\nu = -g\, \partial_\mu A^a_\nu f^{abc} A^b_\mu A^c_\nu$

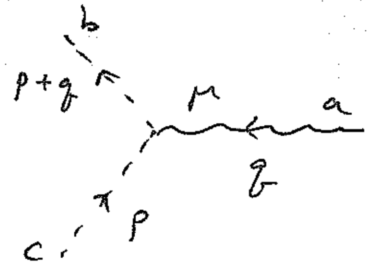
$\Rightarrow$ contracting with $A^d_\rho(y)$: $-g f^{abc} A^b_\mu A^c_\nu \partial_\mu A^a_\nu(x) A^d_\rho(y)$, where the contraction $\sim e^{-ip\cdot(x-y)}$, and $\partial_\mu \to -ip_\mu$

$$\Rightarrow \text{get terms like } \overset{\text{from } iS}{i}\, i g f^{abc} p_\mu = g f^{bac} p_\mu + \ldots$$



$\Rightarrow$ let us summarize all this:

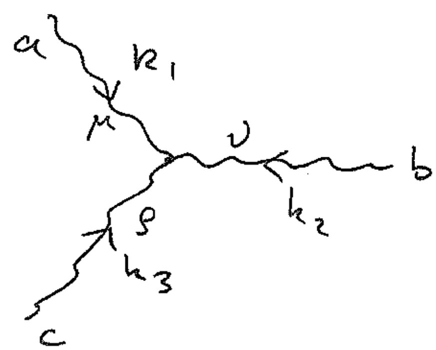
Ghost-gluon Vertex:



$$g\,(p+q)_\mu\, f^{abc}$$

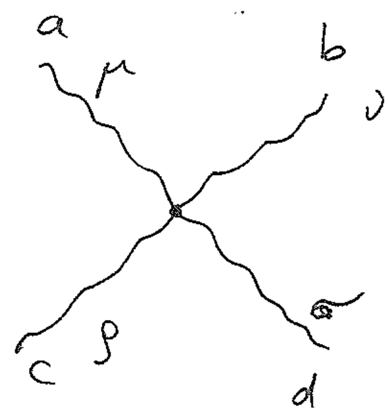
(counter-clockwise)

3-Gluon Vertex:



$$-g f^{abc}\left[(k_1-k_3)_\nu\, g_{\mu\rho} + (k_2-k_1)_\rho\, g_{\mu\nu} + (k_3-k_2)_\mu\, g_{\nu\rho}\right]$$

4-Gluon Vertex:



$$-ig^2\Big[f^{abe} f^{cde}\cdot(g^{\mu\rho}g^{\nu\sigma}-g^{\mu\sigma}g^{\nu\rho}) + f^{ace}f^{bde}(g^{\mu\nu}g^{\rho\sigma}-g^{\mu\sigma}g^{\nu\rho}) + f^{ade}f^{bce}(g^{\mu\nu}g^{\rho\sigma}-g^{\mu\rho}g^{\nu\sigma})\Big].$$

~ same as QED for external fermions, bosons (no external ghosts), internal integrals, "−" for each fermion (or ghost) loop.