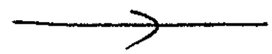
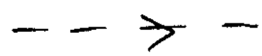
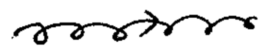
Last time

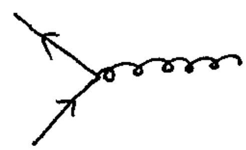
Feynman Rules in QCD (cont'd)

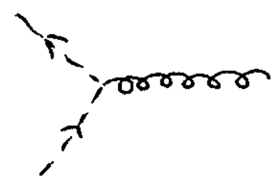
(i) Covariant (Lorenz) gauge ($\partial_\mu A^\mu = 0$)

Quark Propagator 

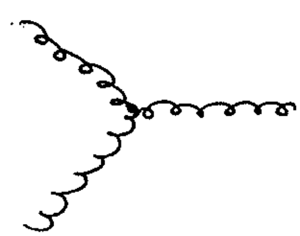
Ghost Propagator 

Gluon Propagator 

Quark-Gluon Vertex 

Ghost-Gluon Vertex 

3-Gluon Vertex



4-Gluon Vertex

(ii) Light-cone gauge

Define light-cone variables: $A^{\pm} = \dfrac{A^0 \pm A^3}{\sqrt{2}}$

(choose a "preferred direction" $\sim x3$)

$A^+ = 0$ gauge is called the light-cone (LC) gauge

Write the gauge condition as

$$\eta \cdot A = 0 \quad \text{with} \quad \eta^- = 1,\ \eta^+ = 0,\ \eta^1 = \eta^2 = 0$$

$$A_\mu B^\mu = A^+ B^- + A^- B^+ - A^1 B^1 - A^2 B^2$$

(check)

$$\eta \cdot A = \underbrace{\eta^+}_{=0} A^- + \underbrace{\eta^-}_{=1} A^+ - \underbrace{\eta^1}_{=0} A^1 - \underbrace{\eta^2}_{=0} A^2 = A^+$$

$\Rightarrow$ there is <u>no ghost</u> in LC gauge!

<u>Feynman rules</u>: the same, but no ghost

$\Rightarrow$ no ghost propagator, no ghost-gluon vertex

$\Rightarrow$ gluon propagator is different:

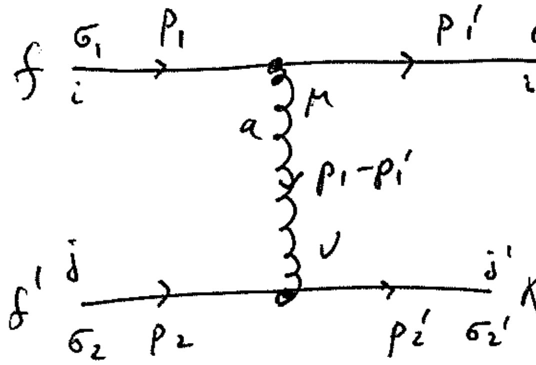
$$= \frac{-i}{k^2 + i\varepsilon}\, \delta^{ab} \left[g_{\mu\nu} - \frac{\eta_\mu k_\nu + \eta_\nu k_\mu}{\eta \cdot k} \right].$$

Example of Using QCD Perturbation Theory:

Quark - Quark scattering, $q_f + q_{f'} \to q_f + q_{f'}$, $f \neq f'$

~ quarks have different flavors

Only one diagram (massless quarks):



color indices

$$iM = \bar{u}_{\sigma_1'}(p_1')\gamma^\mu u_{\sigma_1}(p_1)\ \bar{u}_{\sigma_2'}(p_2')\gamma^\nu u_{\sigma_2}(p_2)$$

$$\cdot\, ig\,(T^a)_{i'i}\ ig\,(T^a)_{j'j}\ \frac{-i g_{\mu\nu}}{(p_1-p_1')^2 + i\varepsilon}$$

Mandelstam $t = (p_1 - p_1')^2$.

$$\Rightarrow \underset{\substack{\uparrow \\ \text{color averaging}}}{\frac{1}{N_c^2}}\ \underset{\substack{\uparrow \\ \text{spin averaging}}}{\frac{1}{4}} \sum_{\substack{\sigma_1,\sigma_1' \\ \sigma_2,\sigma_2'}} |M|^2 = \frac{1}{4} g^4\, \frac{1}{t^2} \cdot \underbrace{\overset{=\frac{1}{2}\delta^{ab}}{tr[T^b T^a]}\ \overset{=\frac{1}{2}\delta^{ab}}{tr[T^b T^a]} \cdot \frac{1}{N_c^2}}_{\text{color factor } = \frac{1}{4}(N_c^2-1)}$$

$$\cdot\, tr[\not{p}_1' \gamma^\mu \not{p}_1 \gamma^\alpha] \cdot tr[\not{p}_2' \gamma_\mu \not{p}_2 \gamma_\alpha] =$$

$$= \frac{1}{4} g^4 \frac{1}{t^2} \cdot \frac{1}{4 N_c^2}(N_c^2 - 1)\ 4\left[p_1'^\mu p_1^\alpha + p_1'^\alpha p_1^\mu - g^{\mu\alpha} p_1 \cdot p_1'\right]$$

\# colors, $N_c = 3$

$$\cdot 4\left[p_{2\mu}' p_{2\alpha} + p_{2\alpha}' p_{2\mu} - g_{\mu\alpha} p_2 \cdot p_2'\right] = \frac{g^4}{t^2}\frac{(N_c^2-1)}{N_c^2} \cdot \Big[2 p_1 \cdot p_2\, p_1' \cdot p_2'$$

$$+ 2 p_1' \cdot p_2\, p_2' \cdot p_1 + \cancel{4\, p_1 \cdot p_1'\, p_2 \cdot p_2'} - \cancel{4\, p_1 \cdot p_1'\, p_2 \cdot p_2'}\Big] = \frac{2g^4}{t^2}\frac{(N_c^2-1)}{N_c^2}$$

$$\cdot \left[p_1 \cdot p_2\, p_1' \cdot p_2' + p_1' \cdot p_2\, p_2' \cdot p_1\right]$$

Assume quarks to be massless:

$$s = (p_1 + p_2)^2 = 2 p_1 \cdot p_2 \quad , \quad p_1 + p_2 = p_1' + p_2'$$

energy-momentum conservation

$$\Rightarrow s = (p_1' + p_2')^2 = 2 p_1' \cdot p_2'$$

$$\Rightarrow p_1 \cdot p_2 \; p_1' \cdot p_2' = \frac{1}{4} s^2$$

$$\left.\begin{aligned} u &= (p_1 - p_2')^2 = -2 p_1 \cdot p_2' \\ u &= (p_2 - p_1')^2 = -2 p_2 \cdot p_1' \end{aligned}\right\} \Rightarrow p_1 \cdot p_2' \; p_2 \cdot p_1' = \frac{u^2}{4}$$

$$\Rightarrow \boxed{\langle |M|^2 \rangle = \frac{g^4}{2t^2} \frac{(N_c^2 - 1)}{N_c^2} \left[s^2 + u^2 \right]} = \frac{N_c^2 - 1}{4 N_c^2} \otimes \text{QED term}$$

$$d\sigma = \frac{1}{2E_1 \, 2E_2 \underbrace{|\vec{v}_1 - \vec{v}_2|}_{=2 \text{ (massless quarks, CMS frame)}}} \frac{d^3 p_1'}{2E_1' (2\pi)^3} \frac{d^3 p_2'}{(2\pi)^3 \, 2E_2'} \cdot (2\pi)^4 \delta^4(p_1 + p_2 - p_1' - p_2')$$

$$\cdot \langle |M|^2 \rangle$$

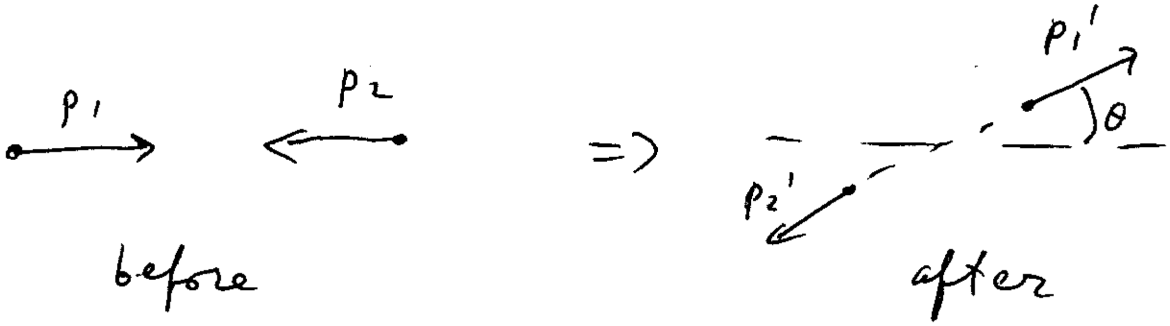
$$= \frac{1}{8 E_1 E_2} \int \frac{\overset{= p_1'^2 \, dp_1' \, d\cos\theta \, d\varphi}{d^3 p_1'}}{2 E_1' (2\pi)^3} \frac{1}{2E_2'} 2\pi \, \delta(E_1 + E_2 - E_1' - E_2')$$

$$\cdot \langle |M|^2 \rangle = \frac{1}{8 E_1 E_2} \frac{1}{4 (2\pi)^3} \int dE_1' \cdot \frac{E_1'}{E_2'} \delta(E_1 + E_2 - E_1' - E_2') \cdot$$

$$\cdot \langle |M|^2 \rangle \cdot 2\pi \, d\cos\theta$$

where $E_2' = E_1'$ (CMS frame) & $M = M(\theta)$ only $\Rightarrow$

$$\Rightarrow d\sigma = \frac{1}{8 E_1^2} \frac{\pi}{16 \pi^2} \cdot d\cos\theta \, \langle |M|^2 \rangle$$



$$t = (p_1 - p_1')^2 \underset{\substack{\uparrow \\ \text{massless} \\ \text{quarks}}}{=} -2 p_1 \cdot p_1' = -2\,(E_1 E_1' - \vec{p}_1 \cdot \vec{p}_1') =$$

$$= -2 E_1^2 (1 - \cos\theta) \leq 0$$

note, $\boxed{t \leq 0}$

$$dt = -2 E_1^2 \sin\theta \, d\theta = 2 E_1^2 \, d\cos\theta$$

$$\Rightarrow d\cos\theta = \frac{dt}{2E_1^2}$$

$$d\sigma = \frac{1}{8E_1^2} \frac{1}{16\pi} \cdot \frac{dt}{2E_1^2} \langle |M|^2 \rangle$$

$$s = (p_1 + p_2)^2 = 2 p_1 \cdot p_2 = 2\left(E_1 E_2 - \vec{p}_1 \cdot \vec{p}_2\right) = 4 E_1^2$$

$$\Rightarrow E_1^2 = s/4 \quad \Rightarrow d\sigma = \frac{1}{2s} \frac{1}{16 \cdot \pi} \, 2 \frac{dt}{s} \langle |M|^2 \rangle$$

$$\Rightarrow \frac{d\sigma}{dt} = \frac{1}{16\pi s^2} \langle |M|^2 \rangle = \frac{1}{16\pi s^2} \frac{g^4}{2t^2} \frac{N_c^2 - 1}{N_c^2} (s^2 + u^2)$$

$$= \Big| \alpha_s = \frac{g^2}{4\pi} \qquad = \frac{\alpha_s^2 \pi}{s^2} \frac{1}{t^2} \frac{N_c^2 - 1}{2N_c^2} (s^2 + u^2)$$

$$\Rightarrow \frac{d\sigma^{f+f' \to f+f'}}{dt} = \frac{\pi \alpha_s^2}{s^2} \frac{N_c^2 - 1}{2N_c^2} \frac{s^2 + u^2}{t^2}$$

Heavy Quark Potential

Imagine two very heavy quarks in vacuum. Can we calculate the force one of them applies on another one? (assuming $Q\bar{Q}$ are in a color-singlet state)

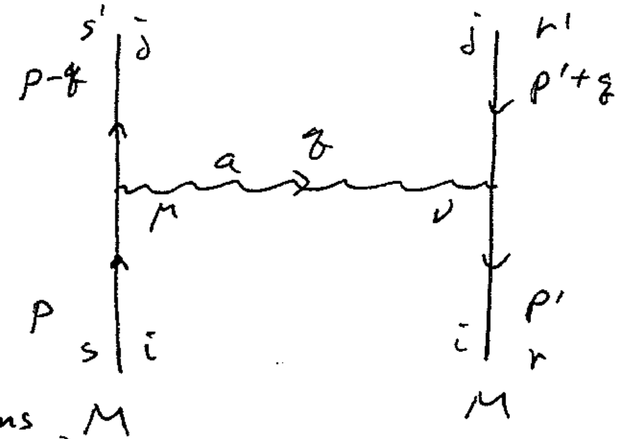
In E&M one has Coulomb potential $V(r) \sim -\frac{\alpha_{EM}}{r}$.

Is it the same in QCD?

Short Distances

at small r the coupling $\alpha_s(1/r^2)$ is small
$\Rightarrow$ can do perturbation theory.

at the lowest order the potential is given by this graph:



The amplitude:

$$iM = \bar{u}_{s'}(p-q)\,\gamma^\mu\, u_s(p)\,\cdot$$

fermion contractions (p. 122 in Peskin)

$$\cdot\,\underbrace{\bar{v}_r(p')\,\gamma^\nu\, v_{r'}(p'+q)}_{\text{need for potential}}\,(-)(ig)^2\,\underbrace{(T^a)_{ji}(T^a)_{ij}}_{\text{color singlet}}\,\frac{-i}{q^2+i\varepsilon}\,\underbrace{g_{\mu\nu}}_{\text{covariant gauge}}$$

$\otimes\ \frac{d^3q\, e^{i\vec{q}\cdot\vec{r}}}{(2\pi)^3}$ $\qquad \frac{1}{N_c} \sim$ average over colors (for potential only)

Quark mass M is very large $\Rightarrow$

$$(p-q)^2 = M^2 \Rightarrow M^2 - 2\,p\cdot q + q^2 = M^2$$

$$\Rightarrow p\cdot q \simeq M\cdot q^0 \quad \Rightarrow M^2 - 2M\cdot q^0 = M^2$$

$$(q^2 \ll p\cdot q) \Rightarrow \quad q^0 = 0 \quad \Rightarrow q^2 = -|\vec{q}|^2.$$

static case

$$\bar{u}_{s'}(p-q)\,\gamma^\mu u_s(p) \simeq g^{\mu 0}\cdot \bar{u}_{s'}(p-q)\,\gamma^0 u_s(p)$$

$$= g^{\mu 0}\, u^+_{s'}(p-q)\, u_s(p) = g^{\mu 0}\cdot 2M\,\delta^{ss'}$$

similarly $\bar{v}_r(p')\gamma^\nu v_{r'}(p'+q) = g^{\nu 0}\, 2M\,\delta^{rr'}$

$$\cancel{i}M = +\cancel{i}g^2 \int \frac{d^3q}{(2\pi)^3}\cdot \underbrace{(2M)^2\delta^{ss'}\delta^{rr'}}_{\text{norm}}\ \frac{1}{\vec{q}^{\,2}}\ \underbrace{tr(T^aT^a)\cdot\frac{1}{N_c}}_{\frac{N_c^2-1}{2N_c}\equiv C_F}$$

To get the potential need to turn d^3q into Fourier transform. Fixing the normalization write ($M \sim -V(q)$)

choose $\vec{r} = r\hat{z}$ in polar coord's

$$\boxed{V(r) = -g^2 C_F \int \frac{d^3q}{(2\pi)^3}\ \frac{e^{i\vec{q}\cdot\vec{r}}}{\vec{q}^{\,2}}} = -g^2 C_F \int_0^\infty \frac{\cancel{q^2}\,dq}{(2\pi)^3}\cdot$$

$$\int_{-1}^{1} d\cos\theta \underbrace{\int_0^{2\pi} d\varphi}_{2\pi}\ \frac{1}{\cancel{q^2}}\,e^{iqr\cos\theta} = -g^2\frac{C_F}{(2\pi)^2}\int_0^\infty dq\,\cdot$$

$$\cdot\,\frac{1}{iqr}\left(e^{iqr} - e^{-iqr}\right) = -\frac{g^2 C_F}{4\pi^2}\frac{1}{ir}\int_0^\infty \frac{dq}{q}\left(e^{iqr} - e^{-iqr}\right)$$

$$= -\frac{g^2}{4\pi^2} C_F \frac{1}{ir} \frac{1}{2} \int_{-\infty}^{\infty} \frac{dq}{q+i\varepsilon} \left(\underbrace{e^{iqr}}_{\substack{\text{close} \\ \text{in upper} \\ \text{half-plane}}} - \underbrace{e^{-iqr}}_{\substack{\text{close in l.h. plane} \\ \Rightarrow \text{ zero}}} \right)$$

$$= -\frac{g^2}{4\pi^{\cancel{2}}} \cdot \frac{C_F}{\cancel{i}r} \cdot \frac{1}{\cancel{2}} \cdot \cancel{2\pi i} = \Big| \alpha_s \equiv \frac{g^2}{4\pi} = -\frac{\alpha_s C_F}{r}$$

$$\Rightarrow \boxed{V_{QCD}(r)\Big|_{r\Lambda \ll 1} \simeq -\frac{\alpha_s C_F}{r}}$$

$\Rightarrow$ attractive Coulomb potential!

just like in QED

$$\Rightarrow C_F = \frac{N_c^2 - 1}{2N_c} = \frac{8}{2 \cdot 3} = \frac{4}{3}$$

$$\Rightarrow \boxed{V_{QCD}(r)\Big|_{r\Lambda \ll 1} \simeq -\frac{4}{3} \frac{\alpha_s}{r}}.$$

$\Rightarrow$ if one drops color factor of $4/3$ and replaces

$\alpha_s \to \alpha_{EM} \Rightarrow$ get QED Coulomb potential

$$V_{QED}(r) = -\frac{\alpha_{EM}}{r}.$$