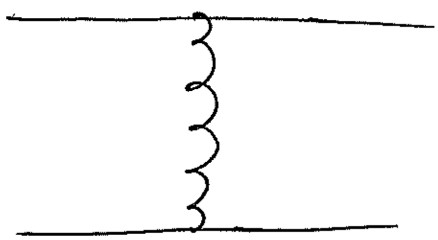
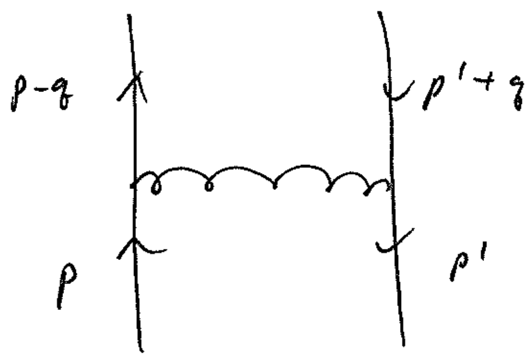
Last time | Calculated a cross-section in QCD perturbation theory: $q_f + q_{f'} \to q_f + q_{f'}$



difference with QED is in color factors!

Heavy Quark Potential:



$$\langle p-q, p'+q | \bar\psi \gamma \psi \; \bar\psi \gamma \psi | p, p' \rangle$$

$$= \langle 0 | d_{p'+q} \, b_{p-q} \, \bar\psi \gamma \psi \; \bar\psi \gamma \psi \; b_p^+ d_{p'}^+ | 0 \rangle$$

$$\psi \sim b + d^+ , \quad \bar\psi \sim b^+ + d$$

untangle $\Rightarrow$ get (-1).

Another "$-$" from the definition of V in terms of M:

$$\boxed{M \sim -V(q)}$$

amplitude potential.

Running Coupling and Asymptotic Freedom

$g \sim$ is the coupling constant

put $m_f = 0$ in $\mathcal{L}_{QCD}$ for simplicity:

$$\left\{ \mathcal{L}_{QCD}^{m_f=0} = \bar{q}^f \, i\gamma^\mu D_\mu q^f - \frac{1}{4} F_{\mu\nu}^a F^{a\mu\nu} \right\}$$

g is the only parameter for such theory.

$\Rightarrow$ When people do perturbation theory, infinities arise:

 $\sim \displaystyle\int^{\mu} \frac{d^4k}{k^4} \sim \ln\mu$ with μ a UV cutoff

- problems are usually in the ultraviolet (UV) where momenta are large
- one has to introduce a UV cutoff μ $\Rightarrow$

$\Rightarrow$ $\mathcal{L}$ & observables would depend on μ:

$$\mathcal{L} = \mathcal{L}(g, \mu), \quad \underset{\substack{\uparrow \\ \text{observable}}}{\mathcal{M}} = \mathcal{M}(g, \mu).$$

$\Rightarrow$ but physics should not be dependent on any cutoff if the theory is consistent $\Rightarrow$

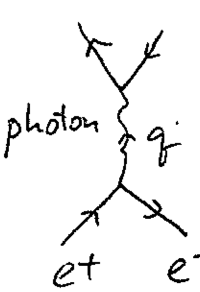
$\Rightarrow$ the only way to make it work is to have g depend on μ $\Rightarrow$ $\mathcal{L} = \mathcal{L}(g_\mu, \mu)$

$\mathcal{M} = \mathcal{M}(g_\mu, \mu)$. $\nwarrow$ re-arrange the expansion in pert. theory to expand in g_μ.

$\Rightarrow$ <u>running coupling</u>: g_μ depends on momentum scale μ.

$\Rightarrow$ imagine an observable $\mathcal{M}$ which depends on a single four-momentum squared: $Q^2 = q_\mu q^\mu$

<u>Example:</u> $e^+e^- \to$ hadrons



$\Rightarrow$ the cross section depends on center of mass energy $Q^2 = q_\mu q^\mu \Rightarrow \sigma = \sigma(Q^2)$

in CM frame $q^\mu = (Q, \vec{0}) \Rightarrow q^2 = Q^2$.

$Q^2\sigma$ is dimensionless; quark masses $= 0$ $\swarrow$ simplicity
electron mass $= 0$.

$\Rightarrow$ in general would have $M = M(Q^2, \alpha_\mu, \mu)$

where $\alpha_\mu = \dfrac{g_\mu^2}{4\pi}$

$\Rightarrow$ assume that M is dimensionless $\Rightarrow M = M\left(\frac{Q^2}{\mu^2}, \alpha_\mu\right)$.

But: no physical observable should depend on μ!

$$\Rightarrow \boxed{\mu^2 \frac{d}{d\mu^2} M\left(\frac{Q^2}{\mu^2}, \alpha_\mu\right) = 0}$$

$$\Rightarrow \left[\mu^2 \frac{\partial}{\partial \mu^2} + \mu^2 \frac{d\alpha_\mu}{d\mu^2} \frac{\partial}{\partial \alpha_\mu}\right] M\left(\frac{Q^2}{\mu^2}, \alpha_\mu\right) = 0$$

Def. Beta-function of QCD: $\beta(\alpha_\mu) = \mu^2 \dfrac{d\alpha_\mu}{d\mu^2}$.

$\beta(\alpha_\mu)$ is dimensionless $\Rightarrow$ can not depend on μ explicitly, μ-dependence comes in through α_μ only!

$$\Rightarrow \boxed{\left[\mu^2 \frac{\partial}{\partial \mu^2} + \beta(\alpha_\mu) \frac{\partial}{\partial \alpha_\mu}\right] M\left(\frac{Q^2}{\mu^2}, \alpha_\mu\right) = 0}$$

renormalization group equation (Callan, Symanzik '70)

$\sim$ tells how things change with the changing momentum scale / distance resolution

$\Rightarrow$ equivalently $\left[-Q^2 \dfrac{\partial}{\partial Q^2} + \beta(\alpha_\mu) \dfrac{\partial}{\partial \alpha_\mu}\right] M\left(\frac{Q^2}{\mu^2}, \alpha_\mu\right) = 0$.

To solve the renormalization group (RG) equation *define* $\rho(\alpha_\mu) = \int_{\alpha_0}^{\alpha_\mu} \frac{d\alpha'}{\beta(\alpha')}$

$\uparrow$ arbitrary cutoff

Def. Running Coupling by:

$$\boxed{\alpha(Q^2) \equiv \rho^{-1}\left(\ln \frac{Q^2}{\mu^2} + \rho(\alpha_\mu)\right)}$$

ρ^{-1} ~ inverse function

$\Rightarrow$ note that

(i) $\alpha(\mu^2) = \alpha_\mu$

(ii) $\left[\mu^2 \frac{\partial}{\partial \mu^2} + \beta(\alpha_\mu)\frac{\partial}{\partial \alpha_\mu}\right]\alpha(Q^2) = 0$

eu (ii) is true because $\left[\mu^2 \frac{\partial}{\partial \mu^2} + \beta(\alpha_\mu)\frac{\partial}{\partial \alpha_\mu}\right]$

$$\cdot\left(\ln \frac{Q^2}{\mu^2} + \rho(\alpha_\mu)\right) = -1 + \beta(\alpha_\mu)\underbrace{\frac{\partial \rho(\alpha_\mu)}{\partial \alpha_\mu}}_{1/\beta(\alpha_\mu) \text{ by definition}} = 0$$

as $M\left(\frac{Q^2}{\mu^2}, \alpha_\mu\right)$ does not depend on μ we can put $\mu = Q$ and get:

$$M\left(\frac{Q^2}{\mu^2}, \alpha_\mu\right) = M\left(\frac{Q^2}{\mu^2}, \alpha(\mu^2)\right) \overset{\mu^2 \to Q^2}{=} M(1, \alpha(Q^2)) = M(\alpha(Q^2))$$

$\Rightarrow$ any M which is a function of $\alpha(Q^2)$ only automatically satisfies RG equation.

$\Rightarrow$ We have shown that running coupling $\alpha(Q^2)$ satisfies RG equation + allows any observable dependent on it to satisfy RG equation.

$\Rightarrow$ let's find $\alpha(Q^2)$: to do this need $\rho(\alpha_\mu)$.

To find $\rho(\alpha_\mu)$ need $\beta(\alpha_\mu)$ ~ the beta-function.

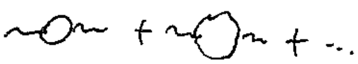
Beta-function has to be found through an explicit (hard) calculation ~ see field theory texts like Peskin.

$\Rightarrow$ in perturbation theory one usually gets:

$$\boxed{\beta(\alpha) = -\beta_2 \alpha^2 - \beta_3 \alpha^3 + \ldots}$$

(perturbative / small coupling α expansion)

in QCD $\quad \beta_2 = \dfrac{11 N_c - 2 N_f}{12\pi}$, $\quad N_c$ ~ # colors

 $\quad N_f$ ~ # flavors

(Politzer '73, Gross & Wilczek '73) $\Leftarrow$ Nobel Prize 2004

~ was probably obtained before by 't Hooft (oral communication)

$\Rightarrow$ it is <u>very important</u> that in QCD $\beta(\alpha) < 0$ ~ beta-function is negative c.f. in QED have $\beta_2^{QED} = -\frac{1}{3\pi}$ such that

$\beta_2^{QED}(\alpha) > 0$.

$\Rightarrow$ why does this matter? Let's do the calculation at small coupling: put $\beta(\alpha) = -\beta_2 \alpha^2$

$$\Rightarrow \rho(\alpha_\mu) = \int_{\alpha_0}^{\alpha_\mu} \frac{d\alpha'}{\beta(\alpha')} = -\frac{1}{\beta_2} \int_{\alpha_0}^{\alpha_\mu} \frac{d\alpha'}{\alpha'^2} = -\frac{1}{\beta_2}\left(-\frac{1}{\alpha'}\right)\Big|_{\alpha_0}^{\alpha_\mu} =$$

$$= \frac{1}{\beta_2}\left(\frac{1}{\alpha_\mu} - \frac{1}{\alpha_0}\right).$$

The inverse function: $\rho(\alpha) = w \Rightarrow \alpha = \rho^{-1}(w)$

$$\Rightarrow \frac{1}{\beta_2}\left(\frac{1}{\alpha} - \frac{1}{\alpha_0}\right) = w \Rightarrow \frac{1}{\alpha} = \frac{1}{\alpha_0} + \beta_2 w \Rightarrow$$

$$\Rightarrow \alpha = \rho^{-1}(w) = \frac{1}{\frac{1}{\alpha_0} + \beta_2 w}.$$

$$\Rightarrow \alpha(Q^2) = \rho^{-1}\left(\ln\frac{Q^2}{\mu^2} + \rho(\alpha_\mu)\right) = \frac{1}{\frac{1}{\alpha_0} + \beta_2\left(\ln\frac{Q^2}{\mu^2} + \rho(\alpha_\mu)\right)}$$

α_0 cancels – not important

$$= \frac{1}{\cancel{\frac{1}{\alpha_0}} + \beta_2\left(\ln\frac{Q^2}{\mu^2} + \frac{1}{\beta_2}\left(\frac{1}{\alpha_\mu} - \cancel{\frac{1}{\alpha_0}}\right)\right)} = \frac{1}{\frac{1}{\alpha_\mu} + \beta_2 \ln\frac{Q^2}{\mu^2}}$$

$$\Rightarrow \boxed{\alpha(Q^2) = \frac{\alpha_\mu}{1 + \alpha_\mu \beta_2 \ln\frac{Q^2}{\mu^2}}}$$

1-loop running coupling in a gauge theory.