Last time: Running Coupling and Asymptotic Freedom

(cont'd)

$$\mathcal{M}(g,\mu) \rightarrow \mathcal{M}(g_\mu,\mu)$$

an observable

coupling must be a ftn of UV cutoff μ.

$$\boxed{\left[\mu^2 \frac{\partial}{\partial \mu^2} + \beta(\alpha_\mu)\frac{\partial}{\partial \alpha_\mu}\right]\mathcal{M}\left(Q^2/\mu^2, \alpha_\mu\right) = 0}$$

Callan-Symanzik equation ~ μ^2-independence of $\mathcal{M}$

Def. $\boxed{\beta(\alpha_\mu) = \mu^2 \frac{d\alpha_\mu}{d\mu^2}}$ ~ beta-function of a field th'y

$$\alpha_\mu = \frac{g_\mu^2}{4\pi}.$$

Def. Running coupling:

$$\boxed{\alpha(Q^2) \equiv \rho^{-1}\left(\ln\frac{Q^2}{\mu^2} + \rho(\alpha_\mu)\right)}$$

where $\rho(\alpha_\mu) = \displaystyle\int_{\alpha_0}^{\alpha_\mu} \frac{d\alpha'}{\beta(\alpha')}$.

$\alpha(Q^2)$ is μ^2-independent.

$$\mathcal{M}\left(Q^2/\mu^2, \alpha_\mu\right) \overset{\mu = Q}{=} \mathcal{M}(1, \alpha(Q^2)) = \mathcal{M}(\alpha(Q^2)) \text{ is also}$$

μ^2-independent

Any ftn. of $\alpha(Q^2)$ is μ^2-independent.

automatically satisfies RG equation.

$\Rightarrow$ We have shown that running coupling $\alpha(Q^2)$ satisfies RG equation + allows any observable dependent on it to satisfy RG equation.

$\Rightarrow$ let's find $\alpha(Q^2)$: to do this need $\rho(\alpha_\mu)$.

To find $\rho(\alpha_\mu)$ need $\beta(\alpha_\mu)$ ~ the beta-function.

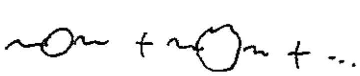
Beta-function has to be found through an explicit (hard) calculation ~ see field theory texts like Peskin.

$\Rightarrow$ in perturbation theory one usually gets:

$$\boxed{\beta(\alpha) = -\beta_2\,\alpha^2 - \beta_3\,\alpha^3 + \ldots}$$

(perturbative / small coupling α expansion)

in QCD $\quad \beta_2 = \dfrac{11\,N_c - 2\,N_f}{12\pi}$, $\quad N_c$ ~ # colors



N_f ~ # flavors

(Politzer '73, Gross & Wilczek '73) $\Leftarrow$ Nobel Prize 2004

~ was probably obtained before by 't Hooft (oral communication)

$\Rightarrow$ it is <u>very important</u> that in QCD $\beta(\alpha) < 0$ ~ beta-function is negative c.f. in QED have $\beta_2^{QED} = -\frac{1}{3\pi}$ such that

$\beta_2^{QED}(\alpha) > 0$.

$\Rightarrow$ why does this matter? Let's do the calculation at small coupling: put $\beta(\alpha) = -\beta_2 \alpha^2$

$$\Rightarrow \rho(\alpha_\mu) = \int_{\alpha_0}^{\alpha_\mu} \frac{d\alpha'}{\beta(\alpha')} = -\frac{1}{\beta_2} \int_{\alpha_0}^{\alpha_\mu} \frac{d\alpha'}{\alpha'^2} = -\frac{1}{\beta_2}\left(-\frac{1}{\alpha'}\right)\Bigg|_{\alpha_0}^{\alpha_\mu} =$$

$$= \frac{1}{\beta_2}\left(\frac{1}{\alpha_\mu} - \frac{1}{\alpha_0}\right).$$

The inverse function: $\rho(\alpha) = w \Rightarrow \alpha = \rho^{-1}(w)$

$$\Rightarrow \frac{1}{\beta_2}\left(\frac{1}{\alpha} - \frac{1}{\alpha_0}\right) = w \Rightarrow \frac{1}{\alpha} = \frac{1}{\alpha_0} + \beta_2 w \Rightarrow$$

$$\Rightarrow \alpha = \rho^{-1}(w) = \frac{1}{\frac{1}{\alpha_0} + \beta_2 w}$$

$$\Rightarrow \alpha(Q^2) = \rho^{-1}\left(\ln\frac{Q^2}{\mu^2} + \rho(\alpha_\mu)\right) = \frac{1}{\frac{1}{\alpha_0} + \beta_2\left(\ln\frac{Q^2}{\mu^2} + \rho(\alpha_\mu)\right)}$$

α_0 cancels - not important

$$= \frac{1}{\cancel{\frac{1}{\alpha_0}} + \beta_2\left(\ln\frac{Q^2}{\mu^2} + \frac{1}{\beta_2}\left(\frac{1}{\alpha_\mu} - \cancel{\frac{1}{\alpha_0}}\right)\right)} = \frac{1}{\frac{1}{\alpha_\mu} + \beta_2 \ln\frac{Q^2}{\mu^2}}$$

$$\Rightarrow \boxed{\alpha(Q^2) = \frac{\alpha_\mu}{1 + \alpha_\mu \beta_2 \ln\frac{Q^2}{\mu^2}}}$$

1-loop running coupling in a gauge theory.

$\Rightarrow$ at large distances / small Q^2 the coupling gets large $\Rightarrow$ pert. th'y breaks down, no one knows what $\alpha_s(Q^2)$ is there.

$\Rightarrow$ when does this happen? write

$$\alpha_s(Q^2) = \frac{\alpha_\mu}{1 + \alpha_\mu \beta_2 \ln \frac{Q^2}{\mu^2}} = \frac{1}{\beta_2 \ln \frac{Q^2}{\Lambda^2} + \underbrace{\frac{1}{\alpha_\mu} - \beta_2 \ln \frac{\mu^2}{\Lambda^2}}_{=0}}$$

define the scale Λ by requiring

$$\Rightarrow \quad \frac{1}{\alpha_\mu} = \beta_2 \ln \frac{\mu^2}{\Lambda^2} \Rightarrow \Lambda^2 = \mu^2 e^{-\frac{1}{\beta_2 \alpha_\mu}} \Rightarrow$$

$\Rightarrow \Lambda^2$ is μ - independent (check).

$$\boxed{\alpha_s(Q^2) = \frac{1}{\beta_2 \ln \frac{Q^2}{\Lambda^2}}}$$

$\Rightarrow$ coupling gets large at $Q^2 \simeq \Lambda^2$.

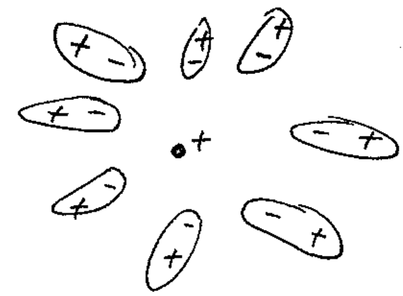
$\Rightarrow \Lambda^2$ is the fundamental parameter in QCD, usually denoted Λ^2_{QCD}.

$\Lambda_{QCD} \simeq 200$ MeV (depends on scale)

(Landau pole: $\alpha_s(\Lambda^2) = \infty \Rightarrow$ Landau thought the theory is inconsistent)

⇒ one can think of running coupling as of the virtual $q\bar{q}$ (or gg) pairs popping out of the vacuum & screening the color charge:

like molecules in a dielectric:



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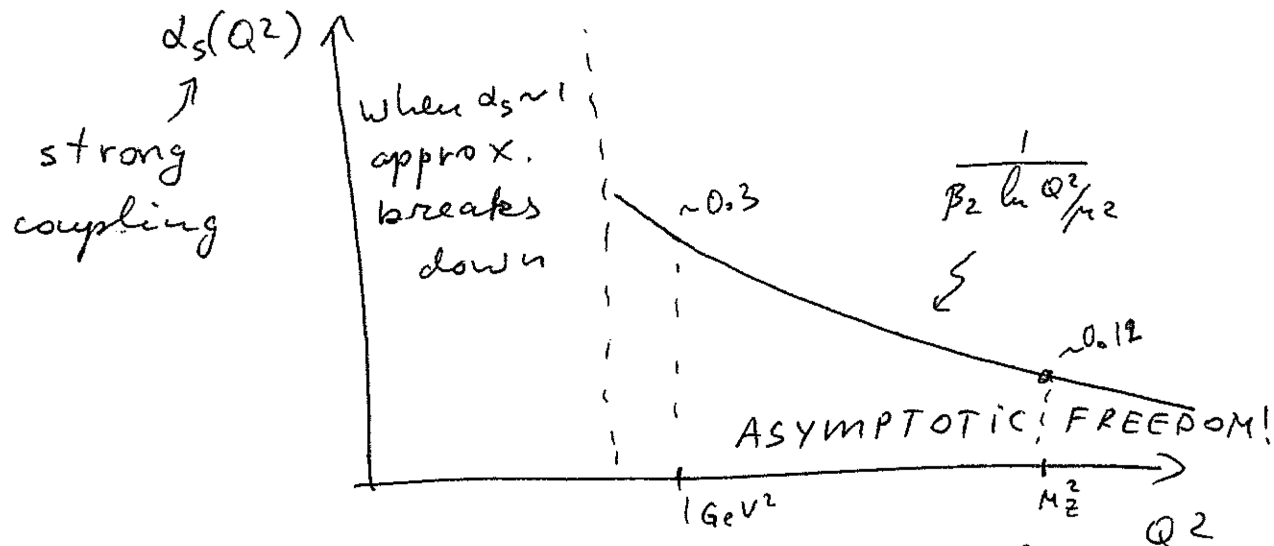
⇒ in QCD $\beta_2 > 0$ ⇒



⇒ at large Q^2 / short distances ($\sim \hbar/Q \sim 1/Q$) the coupling is small!

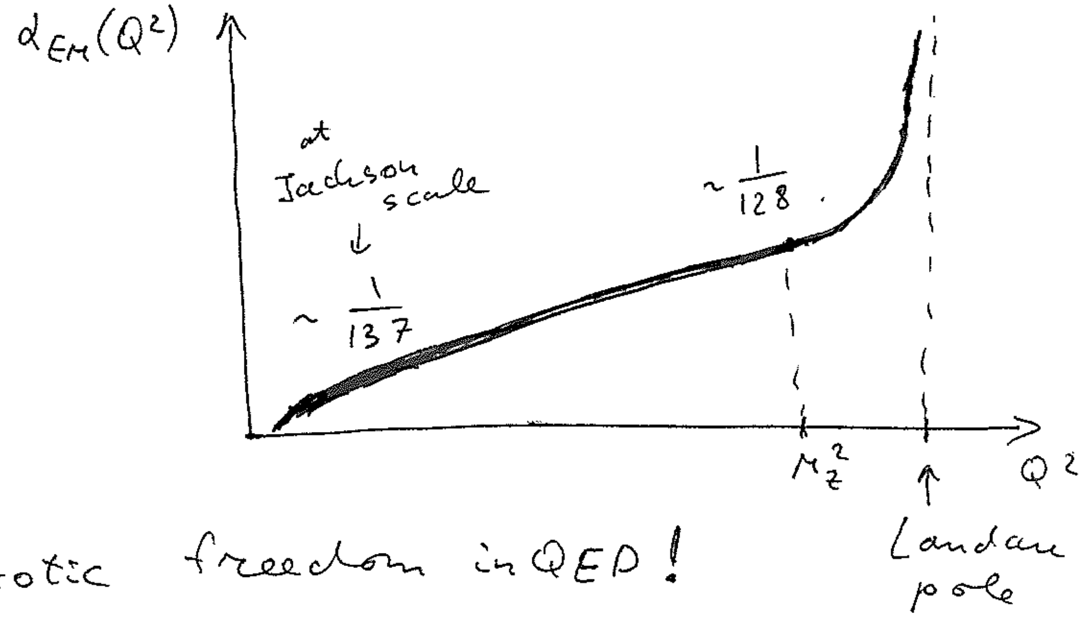
⇒ QCD at short distances is weakly coupled ~ quarks and gluons are asymptotically free! (Politzer, Gross, Wilczek '73)

(see attached plot)

(II) in QED $\beta_2^{QED} < 0 \Rightarrow$

$$\alpha_{EM}(Q^2) = \frac{\alpha_{EM\mu}}{1 + \alpha_{EM\mu} \underbrace{\beta_2^{QED}}_{= -\frac{1}{3\pi}} \ln \frac{Q^2}{\mu^2}} = \frac{\alpha_\mu}{1 - \frac{\alpha_\mu}{3\pi} \ln \frac{Q^2}{\mu^2}}$$

$$\Rightarrow \alpha_{EM}(Q^2) = \frac{\alpha_\mu}{1 + \frac{\alpha_\mu}{3\pi} \ln \frac{\mu^2}{Q^2}}$$ ~ increases with Q^2



$\Rightarrow$ no asymptotic freedom in QED!

$\Rightarrow$ also has a Landau pole, but at large momenta ~ there QED may map onto some more "fundamental" theory, eliminating Landau pole...

$\Rightarrow$ in QCD with massless quarks mesons are massless.

$\Rightarrow$ baryons have a mass. Consider proton (the lightest baryon).

proton mass: M_p ~ dimensionfull quantity.

$M_p = M_p(\alpha_\mu, \mu) = \mu\, f(\alpha_\mu)$ as μ is the only dimension full scale.

$$\mu^2 \frac{d}{d\mu^2} M_p = 0 \Rightarrow \left(\mu^2 \frac{\partial}{\partial \mu^2} + \beta(\alpha_\mu) \frac{\partial}{\partial \alpha_\mu}\right) M_p = 0$$

$$\left(\mu^2 \frac{\partial}{\partial \mu^2} + \beta(\alpha_\mu) \frac{\partial}{\partial \alpha_\mu}\right)\left[\mu f(\alpha_\mu)\right] = 0$$

$$\mu^2 \frac{\partial}{\partial \mu^2}(\mu) = \frac{1}{2}\mu \Rightarrow \left(\frac{1}{2} + \beta \frac{\partial}{\partial \alpha_\mu}\right) f(\alpha_\mu) = 0$$

$$\Rightarrow \frac{d f(\alpha_\mu)}{d\alpha_\mu} = -\frac{1}{2\beta(\alpha_\mu)} f(\alpha_\mu) \Rightarrow \frac{df}{f} = -\frac{d\alpha_\mu}{2\beta(\alpha_\mu)}$$

$$\Rightarrow \ln f(\alpha_\mu) - \ln f(\alpha_0) = -\frac{1}{2}\int_{\alpha_0}^{\alpha_\mu} \frac{d\alpha'}{\beta(\alpha')} = -\frac{1}{2}\rho(\alpha, \alpha_0)$$

$$\Rightarrow f(\alpha_\mu) = f(\alpha_0)\, e^{-\frac{1}{2}\rho(\alpha, \alpha_0)}$$ and the

proton's mass is

$$\boxed{M_p = \mu\ f(\alpha_0)\ e^{-\frac{1}{2}\rho(\alpha_\mu,\alpha_0)}}$$

take $\beta(\alpha) = -\beta_2\alpha^2 \Rightarrow \rho(\alpha) = \int_{\alpha_0}^{\alpha_\mu} \frac{d\alpha'}{\beta(\alpha')} = \frac{1}{\beta_2}\left(\frac{1}{\alpha_\mu} - \frac{1}{\alpha_0}\right)$

$$\Rightarrow M_p = \mu\ f(\alpha_0)\ e^{-\frac{1}{2\beta_2}\left(\frac{1}{\alpha_\mu}-\frac{1}{\alpha_0}\right)}$$

M_p should not depend on α_0 (a cutoff) $\Rightarrow$

$\Rightarrow f(\alpha_0) \propto e^{-\frac{1}{2\beta_2}\frac{1}{\alpha_0}} \Rightarrow$ write $f(\alpha_0) = C_p e^{-\frac{1}{2\beta_2}\frac{1}{\alpha_0}}$

C_p ← constant

$\Rightarrow \boxed{M_p = C_p \cdot \mu \cdot e^{-\frac{1}{2\beta_2}\frac{1}{\alpha_\mu}}}$ ~ non-perturbative dependence on α_μ

$e^{-\frac{1}{x}}$ is a function $\neq$ to its Taylor series

$\Rightarrow$ non-perturbative!

Take $\beta(\alpha) = -\beta_2\alpha^2 - \beta_3\alpha^3 \Rightarrow$

$$\rho(\alpha) = -\frac{1}{\beta_2}\int_{\alpha_0}^{\alpha_\mu} \frac{d\alpha'}{\alpha'^2\left(1+\frac{\beta_3}{\beta_2}\alpha'\right)} = -\frac{1}{\beta_2}\int_{\alpha_0}^{\alpha_\mu}\frac{d\alpha'}{\alpha'^2}\underbrace{\left[1 - \frac{\beta_3}{\beta_2}\alpha' + \dots\right]}_{\text{pert. series}}$$

$$= \frac{1}{\beta_2}\left(\frac{1}{\alpha_\mu} - \frac{1}{\alpha_0}\right) + \frac{\beta_3}{\beta_2^2}\ln\frac{\alpha_\mu}{\alpha_0} + \dots$$

$$\Rightarrow M_p = \mu\ f(\alpha_0)\ e^{-\frac{1}{2}\left[\frac{1}{\beta_2}\left(\frac{1}{\alpha_\mu}-\frac{1}{\alpha_0}\right) + \frac{\beta_3}{\beta_2^2}\ln\left(\frac{\alpha_\mu}{\alpha_0}\right)+\dots\right]}$$

$\Rightarrow$ pick $f(\alpha_0) = C_p \, e^{-\frac{1}{2\beta_2 \alpha_0} - \frac{\beta_3}{2\beta_2^2} \ln \alpha_0}$

$\Rightarrow$ get $$M_p = C_p \, \mu \, \underbrace{e^{-\frac{1}{2\beta_2 \alpha_\mu}}}_{\text{non-analytic ftn.}} (\alpha_\mu)^{-\frac{\beta_3}{2\beta_2^2}} \underbrace{(1 + o(\alpha_\mu))}_{\text{analytic function}}$$

$\Rightarrow$ can not calculate M_p in perturbation theory.

Finally, $M_p = C_p \, \mu \, e^{-\frac{1}{2\beta_2 \alpha_\mu}}$, remember

that $\alpha_\mu = \dfrac{1}{\beta_2 \ln \frac{\mu^2}{\Lambda_{QCD}^2}} \Rightarrow \dfrac{1}{2\beta_2 \alpha_\mu} = \ln \dfrac{\mu}{\Lambda_{QCD}}$

$\Rightarrow M_p = C_p \, \mu \cdot e^{-\ln \frac{\mu}{\Lambda_{QCD}}} = C_p \, \Lambda_{QCD}$

$\Rightarrow$ $M_p \sim \Lambda_{QCD}$ $\sim$ a non-perturbative QCD scale where the coupling α_s is large $\Rightarrow$ can't do perturbation theory there.