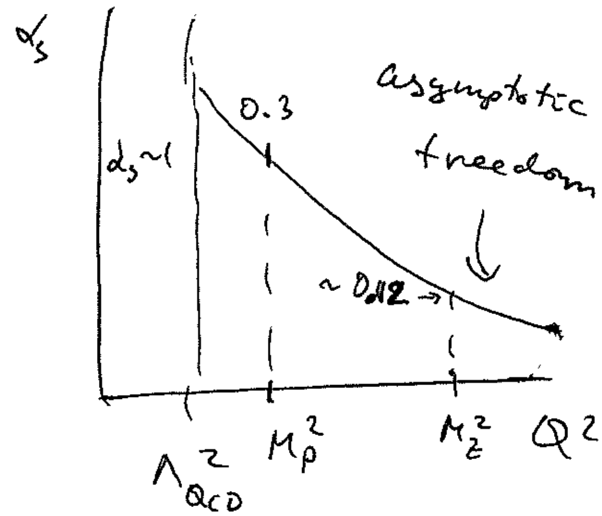
Last time Finished talking about the running coupling. In QCD we observed that the 1-loop running coupling is

$$\boxed{\alpha_s(Q^2) = \frac{\alpha_\mu}{1 + \alpha_\mu \beta_2 \ln \frac{Q^2}{\mu^2}}}$$

$$\beta_2 = \frac{11 N_c - 2 N_f}{12\pi}$$



$$\boxed{\alpha_s(Q^2) = \frac{1}{\beta_2 \ln \frac{Q^2}{\Lambda^2_{QCD}}}}$$

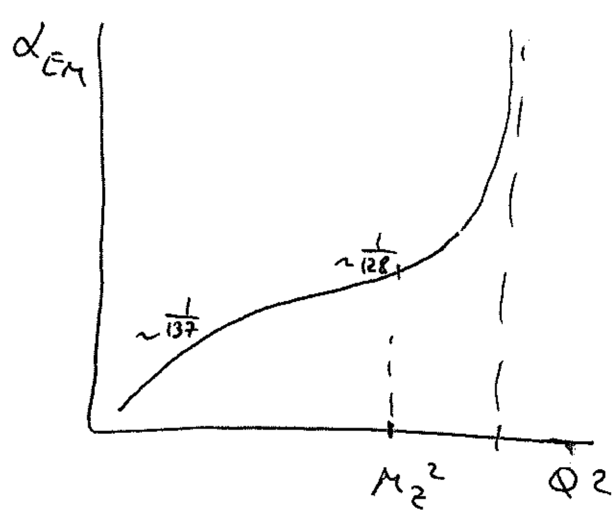
with $\Lambda_{QCD} \simeq 200\,\text{MeV}$ (fundamental scale of QCD)

$Q^2 = \Lambda^2_{QCD}$ ~ Landau pole

In QED:

$$\boxed{\alpha_{EM}(Q^2) = \frac{\alpha_\mu}{1 - \frac{\alpha_\mu}{3\pi} \ln \frac{Q^2}{\mu^2}}}$$

note the sign

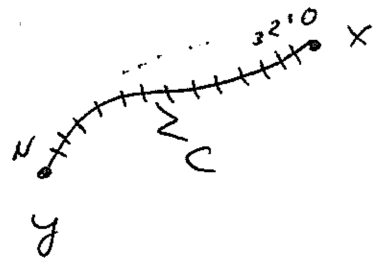


Wilson lines, loops & Heavy Quark Potential

Def. **Wilson line:**

$$\boxed{W_C(x,y) \equiv P_C \exp\left\{ ig \int_y^x dx'_\mu A^\mu(x') \right\}}$$

Def. where a <u>path-ordered exponent</u> is defined as follows. Cut the path C connecting y & x into slices (W_C depends on C!).



Then

$$\boxed{P_c \exp\left\{ ig \int_y^x dx'_\mu A^\mu(x') \right\} \equiv \lim_{N\to\infty} \prod_{i=1}^{N} \left[1 + ig\, \Delta x_i^\mu A_\mu(x_i) \right]}$$

right-most (at N), left-most (at $i=1$)

$$\left(x_0^\mu = x^\mu ,\ x_N^\mu = y^\mu \right), \quad \Delta x_i^\mu = x_{i-1}^\mu - x_i^\mu$$

Under gauge transform $A_\mu(x_i) \to S(x_i) A_\mu(x_i) S^{-1}(x_i) - \frac{i}{g}\left(\partial_\mu S(x_i)\right) S^{-1}(x_i)$

$$\Rightarrow W_c(x,y) \to \prod_{i=1}^{N} \left[1 + ig\, \Delta x_i^\mu \left(S(x_i) A_\mu(x_i) S^{-1}(x_i) - \frac{i}{g}\left(\partial_\mu S(x_i)\right) S^{-1}(x_i) \right) \right] =$$

use
$$S(x_{i-1}) = S(x_i) + \Delta x_i^\mu \partial_\mu S(x_i)$$
and neglect $o(\Delta x^2)$ terms in each factor.

$$= \prod_{i=1}^{N} \left[1 + ig \left(\Delta x_i^\mu \, S(x_{i-1}) \, A_\mu(x_i) \, S^{-1}(x_i) - \frac{i}{g} \left(S(x_{i-1}) - S(x_i) \right) S^{-1}(x_i) \right) \right] = \prod_{i=1}^{N} \left[\cancel{1} + ig \left(\Delta x_i^\mu \, S(x_{i-1}) A_\mu(x_i) S^{-1}(x_i) - \frac{i}{g} S(x_{i-1}) S^{-1}(x_i) + \cancel{\frac{i}{g}} \right) \right] = \prod_{i=1}^{N} S(x_{i-1}) \cdot$$

$$\cdot \left[1 + ig \, \Delta x_i^\mu A_\mu(x_i) \right] S^{-1}(x_i) = S(x) \prod_{i=1}^{N} \left[1 + ig \Delta x_i^\mu A_\mu(x_i) \right] \cdot$$

$$\cdot S^{-1}(y) = S(x) \, W_c(x,y) \, S^{-1}(y).$$

$$\Rightarrow \boxed{W_c(x,y) \longrightarrow S(x) W_c(x,y) \, S^{-1}(y)}$$

Def. Wilson loop:

$tr[W_c(x,x)]$ is called a <u>Wilson loop</u>.

(K. Wilson, '74?)

Under gauge transformation

$$tr[W_c(x,x)] \to tr[S(x) W_c(x,x) S^{-1}(x)] = tr[W_c(x,x)]$$

invariant! Wilson loop is gauge-invariant!

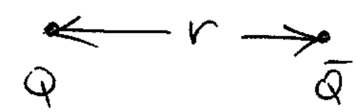
uses:

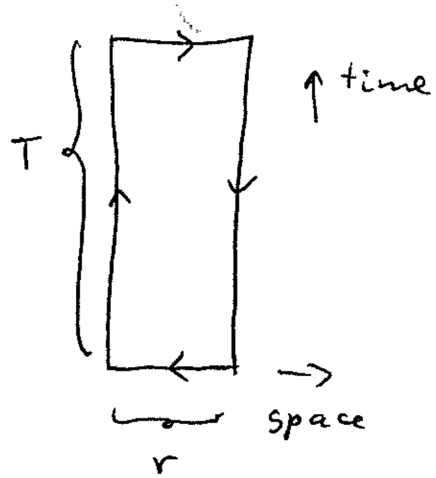
$\Rightarrow$ Wilson line represents quark propagator when one can neglect recoil. This works in high energy scattering and for static heavy quarks.

$\Rightarrow$ Wilson lines form links which can be used to define QCD action on the lattice for numerical simulations.

Heavy Quark Potential:

Suppose one wants to find heavy $Q\bar{Q}$ potential in QCD. How does one define the potential $V(r)$ in a gauge-invariant way?





Take a Wilson loop defined as shown.

$$\langle W \rangle \Big|_{T \to \infty} \simeq e^{-i T V(r)}$$

neglect interaction with gauge links (it does not scale with T to the same degree)

$$V(r) = \lim_{T \to \infty} \left[\frac{i}{T} \ln \langle W \rangle \right]$$

~ can calculate numerically on the lattice

Note that, since Feynman path integral time-ordering operators, one can write

$$\boxed{tr\left[W_c(x,x)\right] = \frac{\int \mathcal{D}A_\mu \, e^{iS[A_\mu]} \cdot e^{ig\int j^a_\mu(x)\, A^{a\mu}(x)\, d^4x}}{\int \mathcal{D}A_\mu \, e^{iS[A_\mu]}}}$$

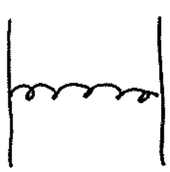
where $j^a_\mu(x)$ is some external current, which is non-zero only along the contour C.

---

r is the only scale in $V(r) \Rightarrow \alpha_s = \alpha_s(1/r^2)$

if $r \ll \frac{1}{\Lambda_{QCD}} \Rightarrow \alpha_s\left(\frac{1}{r^2}\right) \ll 1 \Rightarrow$ can use perturbative QCD

The potential is (see pp. 38-40 of these notes)



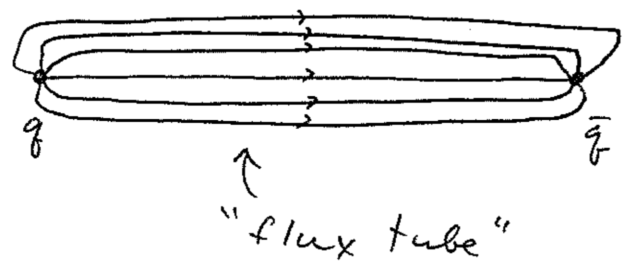
$$\boxed{V(r)\Big|_{r \ll \frac{1}{\Lambda_{QCD}}} \simeq -\frac{\alpha_s C_F}{r} = -\frac{4}{3}\frac{\alpha_s}{r}}$$

$\Rightarrow$ this is a Coulomb-like potential, similar to classical E&M.

<u>Longer Distances</u>: $r\Lambda_{QCD} \gtrsim 1 \Rightarrow$

$\alpha_s = \alpha_s\left(\frac{1}{r^2}\right) \sim \alpha_s(\Lambda_{QCD}^2) \sim 1 \Rightarrow$ perturbative approach breaks down as α_s is not small anymore!

<u>Qualitative</u> picture of what happens: draw force lines as:



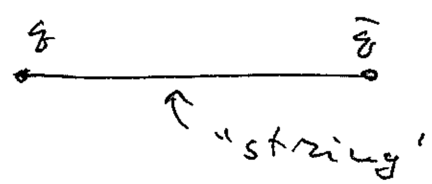
$\sim$ constant force in-between, inside the flux tube

$\Rightarrow \quad V(r) \propto \underset{\text{force}}{F} \cdot r \quad \Rightarrow \quad V(r)\Big|_{r\Lambda_{QCD} \gg 1} \simeq \sigma r$

dimensions of σ ~ mass squared, $\sigma \simeq \Lambda_{QCD}^2$.

$\Rightarrow$ think of a flux tube as a relativistic string: σ is <u>string tension</u>:

$$\sigma \simeq 1 \frac{GeV}{fm} \approx \frac{1}{5} GeV^2$$



Relativistic particle: the action is proportional to proper time τ, such that

$$S_{particle} = -mc^2 \int d\tau .$$

Relativistic string: the action is proportional to "proper area" of a world-sheet:

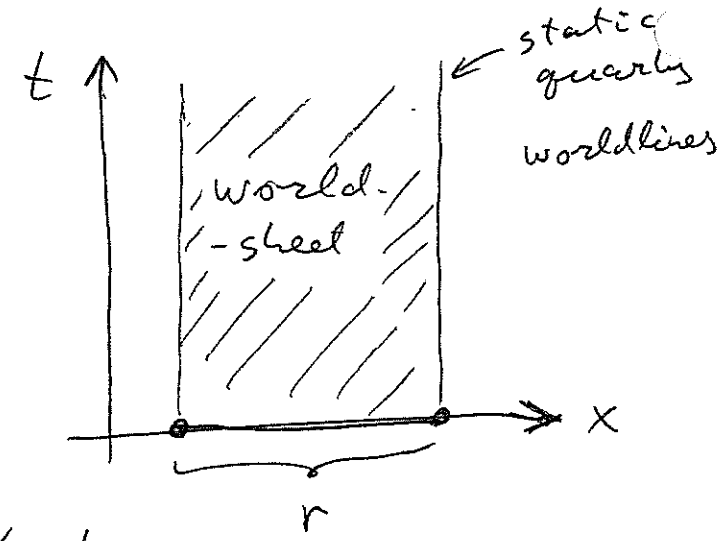
$$S_{string} = -\sigma \cdot (\text{Area}).$$

(put $c=1$ for simplicity).

Consider a static string between 2 quarks:

to find classical configuration need to extremize the action

S_{string} => minimize



the area of string worldsheet

=> obviously min. is achieved for straight string with the action $S^{classical}_{string} = -\sigma \cdot \int dt \cdot \int_0^r dx$

$$= -\sigma \int dt \cdot r = \int dt \cdot L = \int dt \, (\underset{\substack{\parallel \\ 0}}{K} - V(r)) = \int dt \, [-V(r)]$$

as no motion

Lattice QCD: data points

perturbative QCD: solid lines + band.