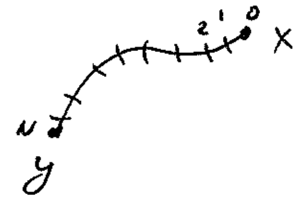
Last time: Wilson lines, loops & Heavy Quark Potential (cont'd)

Wilson line

$$\boxed{W_c(x,y) = P_c \exp\left\{ ig \int_y^x dx'_\mu \, A^\mu(x') \right\}}$$



$$\boxed{W_c(x,y) = \lim_{N\to\infty} \prod_{i=1}^{N} \left[1 + ig \, \Delta x_i^\mu A_\mu(x_i) \right]}$$

$P_c \exp$ ~ path-ordered exponential

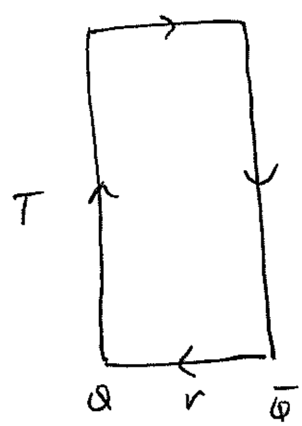
Under gauge transformations:

$$\boxed{W_c(x,y) \rightarrow S(x) \, W_c(x,y) \, S^{-1}(y)} \Rightarrow$$

Wilson loops: $\mathrm{tr}\left[W_c(x,x) \right]$ gauge-invariant

__Heavy Quark Potential:__

$$\boxed{V(r) = \lim_{T\to\infty} \left[\frac{i}{T} \ln \langle W(r,T) \rangle \right]}$$



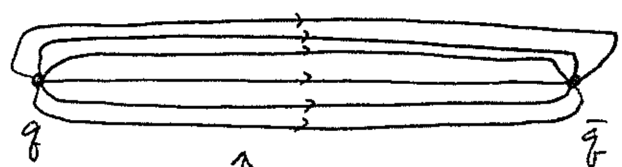
$r \ll \frac{1}{\Lambda_{QCD}} \Rightarrow V(r) \simeq -\frac{\alpha_s C_F}{r}$

$r \gg \frac{1}{\Lambda_{QCD}} \Rightarrow V(r) \simeq \sigma r \,, \quad \sigma \simeq 1 \frac{\mathrm{GeV}}{\mathrm{fm}}$ ~ string tension

<u>Longer Distances</u>: $r\Lambda_{QCD} \gtrsim 1 \Rightarrow$

$\alpha_s = \alpha_s\left(\frac{1}{r^2}\right) \sim \alpha_s(\Lambda_{QCD}^2) \sim 1 \Rightarrow$ perturbative approach breaks down as α_s is not small anymore!

<u>Qualitative</u> picture of what happens: draw force lines as:



"flux tube"

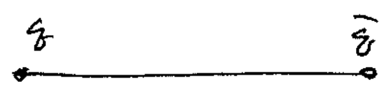
$\sim$ constant force in-between, inside the flux tube

$$\Rightarrow \quad V(r) \propto \underset{\text{force}}{F} \cdot r \quad \Rightarrow \quad V(r)\Big|_{r\Lambda_{QCD} \gg 1} \simeq \sigma r$$

dimensions of σ ~ mass squared, $\sigma \simeq \Lambda_{QCD}^2$.

$\Rightarrow$ think of a flux tube as a relativistic string: σ is <u>string tension</u>:

$$\sigma \simeq 1\,\frac{\text{GeV}}{\text{fm}} \simeq \frac{1}{5}\,\text{GeV}^2$$



"string"

Relativistic particle: the action is proportional to proper time τ, such that

$$S_{\text{particle}} = -mc^2 \int d\tau .$$

Relativistic string: the action is proportional to "proper area" of a world-sheet:

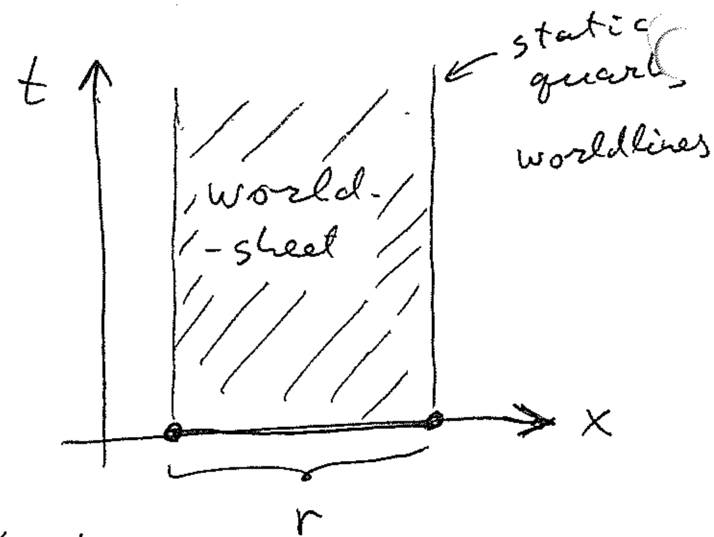
$$S_{\text{string}} = -\sigma \cdot (\text{Area}).$$

(put $c=1$ for simplicity).

Consider a static string between 2 quarks:

to find classical configuration need to extremize the action

S_{string} => minimize



the area of string worldsheet

=> obviously min. is achieved for straight string with the action $S^{\text{classical}}_{\text{string}} = -\sigma \cdot \int dt \cdot \int_0^r dx$

$$= -\sigma \int dt \cdot r = \int dt \cdot L = \int dt \, (\underset{\substack{\parallel \\ 0}}{K} - V(r)) = \int dt \, [-V(r)]$$

as no motion

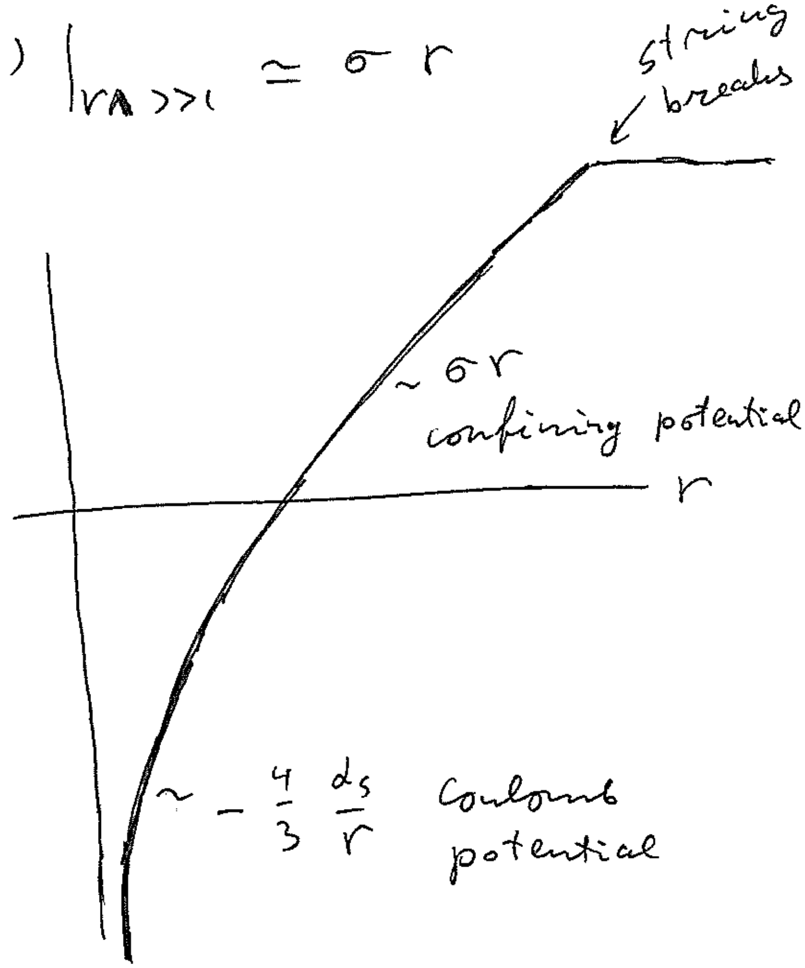
$\Rightarrow \; V(r) = \sigma r$ as desired!

(note the difference from non-relativistic string in classical mechanics which has $V(r) \sim \frac{1}{2} k r^2 \Rightarrow$ force $= kr$)

$\Rightarrow$ the attractive force is <u>constant</u>: $F = \sigma$.

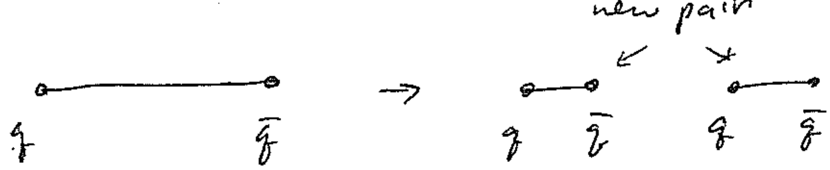
$\Rightarrow$ We know that
$$\begin{cases} V(r)\big|_{r\Lambda \ll 1} \simeq -\frac{4}{3}\frac{\alpha_s}{r} \\ V(r)\big|_{r\Lambda \gg 1} \simeq \sigma r \end{cases}$$

The full potential is:
(see attached lattice data handout)



Linear potential is <u>confining</u>: quarks can not escape.

If string breaks $\Rightarrow$
$\Rightarrow$ get $q\bar{q}$ pair out of the vacuum:



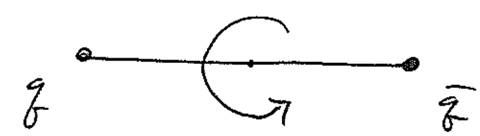
Good interpolation:

$$\boxed{V(r) \simeq -\frac{4}{3}\frac{\alpha_s}{r} + \sigma r}$$

"Cornell potential".

String model works amazingly well: think of $q\bar{q}$ state as a meson. If the meson has spin $\Rightarrow$ think of an ultra-relativistic rotating string:

d ~ string length



if q & $\bar{q}$ rotate with velocity $= 1$ (UR quarks) $\Rightarrow$ $v = \frac{r}{d/2} = \frac{2r}{d}$.

r ~ distance from string element to rot. center

v ~ velocity of string element.

$$\Rightarrow \quad M = \int \frac{dm}{\sqrt{1-v^2}} = 2\int_0^{d/2} \frac{\sigma\, dr}{\sqrt{1-v^2}} = 2\sigma \cdot \int_0^{d/2} \frac{dr}{\sqrt{1-\left(\frac{2r}{d}\right)^2}} = 2\sigma \cdot \frac{d}{2} \cdot \underbrace{\int_0^1 \frac{d\xi}{\sqrt{1-\xi^2}}}_{\substack{\pi/2 \\ \shortparallel \\ (\arcsin \xi)\big|_0^1}} = \frac{\pi}{2}\sigma d.$$

The angular momentum (meson's spin)

is $J = \int \frac{r v\, dm}{\sqrt{1-v^2}} = 2\sigma \int_0^{d/2} \frac{r v\, dr}{\sqrt{1-v^2}} =$

$$= 2\sigma \int_0^{d/2} \frac{dr \cdot (2r/d) \cdot r}{\sqrt{1-\left(\frac{2r}{d}\right)^2}} = 2\sigma \left(\frac{d}{2}\right)^2 \underbrace{\int_0^1 d\xi \frac{\xi^2}{\sqrt{1-\xi^2}}}_{\pi/4} =$$

$$= \frac{\sigma d^2}{2} \cdot \frac{\pi}{4} = \frac{\pi \cdot \sigma d^2}{8} .$$

$\Rightarrow$ meson mass $\boxed{M = \frac{\pi}{2} \sigma d}$ Gasiorowicz & Rosner '81

meson spin $\boxed{J = \frac{\pi \sigma d^2}{8}}$

$$\Rightarrow \quad J = \frac{\pi}{8} \sigma \cdot \left(\frac{2M}{\pi\sigma}\right)^2 = \frac{1}{2\pi\sigma} M^2$$

$\Rightarrow$ $J = \frac{1}{2\pi\sigma} M^2$ is an example of a Regge trajectory

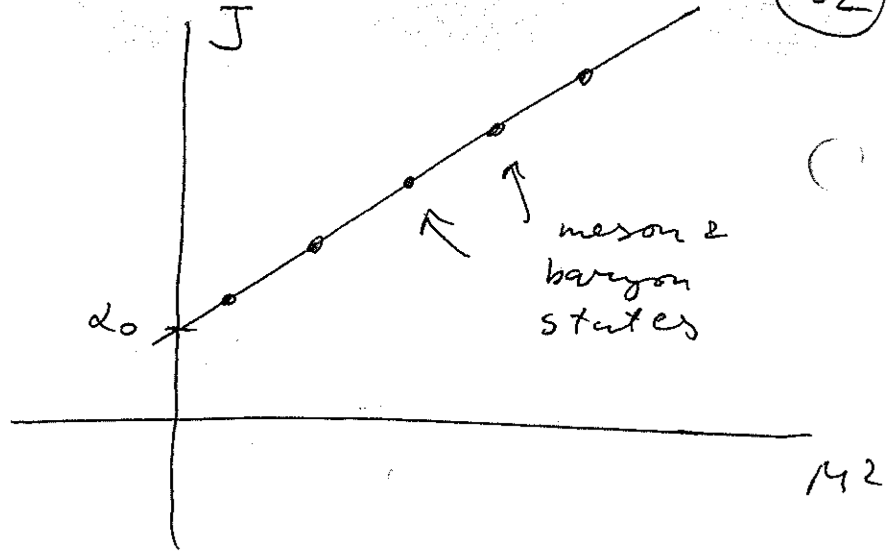
In general, on the basis of phenomenological evidence, people noticed that

$\boxed{J = \alpha_0 + \alpha' M_J^2}$ Chew & Frautschi '61

α_0 ~ intercept

α' ~ slope

Regge trajectory:



we got $\alpha' = \dfrac{1}{2\pi\sigma}$

or $\sigma = \dfrac{1}{2\pi\alpha'}$.

using $\sigma = 1\ \text{GeV/fm}$

$$\alpha' = \frac{1}{2\pi\sigma} \approx \frac{5}{2\pi}\ \text{GeV}^{-2}$$

experimentally $\alpha' \simeq 0.25\ \text{GeV}^{-2}$.

⇒ successes of string approximation to strong interaction data led to proposal of string theory as the theory of strong interactions in the '60's.

⇒ that idea was killed by $e^+e^- \to$ hadrons & (mostly) DIS data & string theory moved on to gravity in '84.

Quark Symmetries Revisited.

⇒ Isospin symmetry: we had an isospin operator $\vec{I}$ which was like angular momentum operator in 3d isospin fictitious space; as angular momentum it satisfied:

$$[I_a, I_b] = i\,\varepsilon_{abc}\, I_c$$

Compare with SU(2): group elements were $e^{i\vec{\alpha}\cdot\vec{J}}$ with $\vec{J} = \frac{1}{2}\vec{\sigma}$. We had the Lie algebra for generators:

$$[J_i, J_j] = i\,\varepsilon_{ijk}\, J_k .$$

⇒ isospin symmetry is an <u>SU(2) symmetry</u>!

Fundamental representation $\square$ of SU(2) is 2

⇒ a doublet ⇒ we saw a lot of isospin doublets

$$\begin{pmatrix} p \\ n \end{pmatrix}, \begin{pmatrix} K^+ \\ K^0 \end{pmatrix}, \begin{pmatrix} \bar{K}^0 \\ K^- \end{pmatrix}, \ldots$$

$$2 \otimes \bar{2} = \square \otimes \square = \underbrace{\begin{smallmatrix}\square\\\square\end{smallmatrix}}_{= \mathbb{1}\ \text{(singlet)}} \oplus \underbrace{\square\square}_{= 3\ \text{(triplet)}} = 1 \oplus 3$$

⇒ can have isospin 3 ~ triplets: $\begin{pmatrix} \pi^+ \\ \pi^0 \\ \pi^- \end{pmatrix}, \begin{pmatrix} \rho^+ \\ \rho^0 \\ \rho^- \end{pmatrix}, \begin{pmatrix} \Sigma^+ \\ \Sigma^0 \\ \Sigma^- \end{pmatrix}, \ldots$

$\Rightarrow$ can also have iso-singlets: $\eta, \omega, \phi, \Lambda, \ldots$

Is this a symmetry of the quark Lagrangian that we wrote? Look at 2 flavors:

$$q(x) = \begin{pmatrix} u(x) \\ d(x) \end{pmatrix} \; ; \quad m = \begin{pmatrix} m_u & 0 \\ 0 & m_d \end{pmatrix} \quad \Rightarrow$$

$$\Rightarrow \quad \mathcal{L}^{N_f=2}_{quarks} = \bar{q}\left(i\gamma\cdot\partial - m\right) q$$

First put $m=0 \Rightarrow \mathcal{L}_0 = \bar{q}^f \, i\gamma\cdot\partial \, q^f$

$\Rightarrow$ SU(2) flavor transformation would be

$$e^{i\vec{\alpha}\cdot\frac{\vec{\sigma}}{2}} \quad \Rightarrow \quad q \to q' = e^{i\vec{\alpha}\cdot\frac{\vec{\sigma}}{2}} q \;, \quad \vec{\alpha} \sim \text{const} \; (x\text{-indep.})$$

$\Rightarrow \bar{q} \to \bar{q}' = \bar{q} e^{-i\vec{\alpha}\frac{\vec{\sigma}}{2}}$ $\Rightarrow$ $\mathcal{L}_0$ is invariant:

$$\bar{q}\, i\gamma\cdot\partial q \to \bar{q}'\, i\gamma\cdot\partial q' = \bar{q}\, e^{-i\vec{\alpha}\frac{\vec{\sigma}}{2}}\, i\gamma\cdot\partial\, e^{i\vec{\alpha}\frac{\vec{\sigma}}{2}} q =$$

$$= \bar{q}\, i\gamma\cdot\partial q.$$

What about the mass term?

$$\bar{q} m q \to \bar{q}' m q' = \bar{q}\, e^{-i\vec{\alpha}\cdot\frac{\vec{\sigma}}{2}}\, m\, e^{i\vec{\alpha}\frac{\vec{\sigma}}{2}} q$$

Write $m = \begin{pmatrix} \frac{m_u+m_d}{2} & 0 \\ 0 & \frac{m_u+m_d}{2} \end{pmatrix} + \begin{pmatrix} \frac{m_u-m_d}{2} & 0 \\ 0 & -\frac{m_u-m_d}{2} \end{pmatrix} \Rightarrow$

$\Rightarrow\ m = \frac{m_u+m_d}{2}\,\mathbb{1} + \frac{m_u-m_d}{2}\,\sigma^3$

$\Rightarrow\ \bar{q}'\, m\, q' = \bar{q}\,\frac{m_u+m_d}{2}\,q + \frac{m_u-m_d}{2}\,\bar{q}\,\underbrace{e^{-i\vec{\alpha}\cdot\frac{\vec{\sigma}}{2}}\,\sigma^3\, e^{i\vec{\alpha}\cdot\frac{\vec{\sigma}}{2}}}_{\neq\ \sigma^3}\,q$

$\Rightarrow$ if $m_u = m_d$ $\Rightarrow$ get exact $SU(2)$ flavor symmetry (global $SU(2)$ symmetry ~ $\vec{\alpha}$ is independent of x^μ)

as $m_u \neq m_d$ by a little bit $\Rightarrow$ $SU(2)$ flavor is (slightly) broken. ($\Rightarrow$ hadron masses are different)

$\Rightarrow$ in reality the symmetry group is much larger!

~ $SU(2)_R \times SU(2)_L$ ~ more on this later.
(for massless quarks)

$\Rightarrow$ Now, put the strange quark back in:

$$q = \begin{pmatrix} u(x) \\ d(x) \\ s(x) \end{pmatrix}, \quad m = \begin{pmatrix} m_u & 0 & 0 \\ 0 & m_d & 0 \\ 0 & 0 & m_s \end{pmatrix}$$

$\Rightarrow$ $\mathcal{L}^{N_f=3}_{\text{quarks}} = \bar{q}\,(i\gamma\cdot\partial - m)\,q$ again.

$\Rightarrow$ one can check that if $m_u = m_d = m_s$ then $\mathcal{L}$ is invariant under $SU(3)$ flavor transform:

$q \rightarrow q' = e^{i\vec{\alpha}\cdot\vec{T}}\,q$, $T^a = \frac{1}{2}\lambda^a$, λ^a ~ Gell-Mann matrices

$a = 1, 2, \ldots, 8.$

$\Rightarrow$ as $m_u \neq m_d \neq m_s$ SU(3) is not an exact flavor symmetry.

Now, let's look at mesons: $\bar{q}q$ ~ states

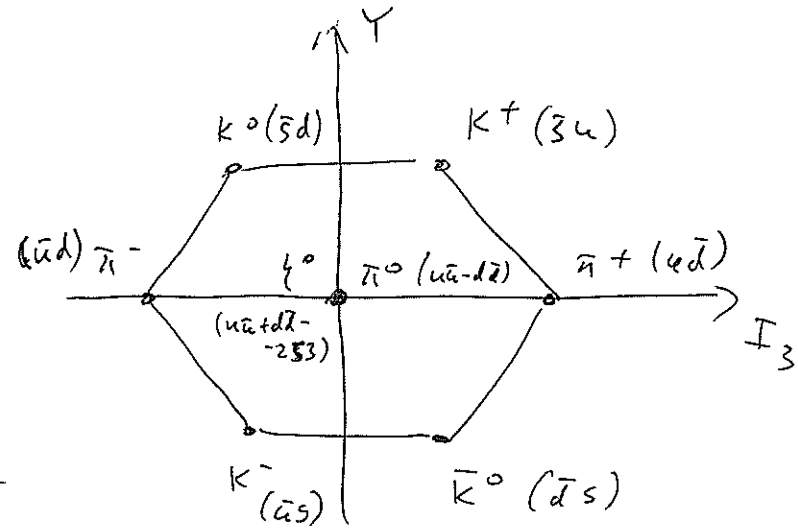
$\Rightarrow 3 \otimes \bar{3} = 1 \oplus 8$ $\Rightarrow$ there should be a flavor octet and singlet:

pseudoscalar mesons $\pi^+, \pi^-, \pi^0, \eta^0, K^+, K^0, \bar{K}^0, K^-$ form flavor-octet!

"The Eightfold Way"!

η' ~ flavor singlet! $\sim (\bar{u}u + \bar{d}d + \bar{s}s)\frac{1}{\sqrt{3}}$.

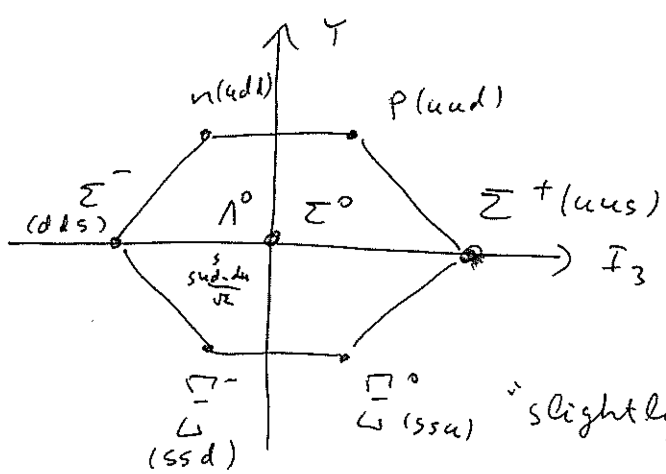
vector mesons ~ the same story!



What about baryons? qqq-states $\Rightarrow$

$3 \otimes 3 \otimes 3 = 1 \oplus 8 \oplus 8 \oplus 10$

$\frac{1}{2}^+$ baryons: $p, n, \Sigma^+, \Sigma^-, \Sigma^0, \Lambda^0, \Xi^0, \Xi^-$ ~ form an octet!



baryon decuplet ~ that's the 10!

$\Rightarrow$ as SU(3) flavor is not exact, all masses are "slightly" different ~ broken symmetry!

$(m_{\Xi^0} = 1315\ \text{MeV},\ m_p = 938\ \text{MeV})$