Wilson line

\[ W_c(x, y) = P_c \exp \left\{ i g \int_{x'}^x A^\mu(x') dx' \right\} \]

\[ W_c(x, y) = \lim_{N \to \infty} \prod_{i=1}^N \left[ 1 + i g A_i^\mu(x_i) \right] \]

\( P_c \exp \) ~ path-ordered exponential

Under gauge transformations:

\[ W_c(x, y) \rightarrow S(x) W_c(x, y) S^{-1}(y) \]

Wilson loops:

\[ \text{tr}[W_c(x, x)] \]

gauge-invariant

Heavy Quark Potential:

\[ V(r) = \lim_{T \to \infty} \left[ \frac{i}{T} \ln \left\langle W(r, r) \right\rangle \right] \]

For small distances:

\[ r \ll \frac{1}{\Lambda_{\overline{QCD}}} \Rightarrow V(r) = - \frac{\alpha_s c_F}{r} \]

For large distances:

\[ r \gg \frac{1}{\Lambda_{\overline{QCD}}} \Rightarrow V(r) \approx \sigma r, \quad \sigma \approx \frac{1 \text{ GeV}}{\text{fm}} \text{ ~ string tension} \]
Longer Distances: \( r / \Lambda QCD \gg 1 \rightarrow \)

\( d_s = d_s (\frac{1}{r^2}) \sim d_s (\Lambda QCD)^2 r \rightarrow \) perturbative

approach breaks down as \( d_s \) is not small

anymore!

Qualitative picture of what happens: draw force

d lines as:

\[ \text{"flux tube"} \]

\( \text{a constant force in-between, inside the flux tube} \)

\( \Rightarrow V(r) \propto F \cdot r \Rightarrow \)

\( V(r) = \frac{\sigma}{r} \)

\( \Lambda QCD \gg 1 \)

dimensions of \( \sigma \sim \) mass squared, \( \sigma = \Lambda QCD^2 \)

\( \Rightarrow \) think of a flux tube as a relativistic

string: \( \sigma \) is string tension:

\[ \sigma \sim 1 \text{ GeV}^2 \text{ fm}^{-1} \approx 1 \text{ GeV}^2 / 5 \]

\[ \text{"string"} \]
Relativistic particle: the action is proportional to proper time $\tau$, such that
\[ S_{\text{particle}} = -mc^2 \int \sigma d\tau. \]

Relativistic string: the action is proportional to "proper area" of a world-surface:
\[ S_{\text{string}} = -\sigma \int (\text{Area}). \]

(put $c = 1$ for simplicity).

Consider a static string between 2 quarks.

To find classical configuration need to extremize the action
\[ S_{\text{string}} \Rightarrow \text{minimize} \]
the area of string worldsheet
\[ \Rightarrow \text{obviously min. is achieved for straight string with the action } S_{\text{string}}^{\text{classical}} = -\sigma \int dt \int dx \]
\[ = -\sigma \int dt \int dx \Rightarrow \text{L} = \int dt \left( K - V(r) \right) = \int dt \left[ F(r) \right] \]
as no motion
\[
V(r) = 5r
\]
as desired!

(note the difference from non-relativistic string in classical mechanics which has
\[
V(r) \sim \frac{1}{2} kr^2 \Rightarrow \text{force} = kr
\]

\Rightarrow \text{the attractive force is constant: } F = 6.

\Rightarrow \text{We know that } \left\{ \begin{array}{l}
V(r) \mid_{r \ll 1} = -\frac{4}{3} \frac{dS}{r} \\
V(r) \mid_{r \gg 1} = 5r
\end{array} \right.

The full potential is:

[see attached lattice V(r) data handout]

Linear potential is

**confining**: quarks cannot escape.

If string breaks \Rightarrow

\Rightarrow get q \bar{q} pair out of the vacuum:

\[\begin{array}{c}
\bar{q} \\
q
\end{array}\]
Good interpolation:

\[ V(r) = -\frac{4}{3} \frac{ds}{r} + \sigma \cdot r \]

"Cornell potential".

String model works amazingly well: think of $q \bar{q}$ state as a meson. If the meson has spin $\Rightarrow$ think of an ultra-relativistic rotating string:

- $d$ = string length
- if $q$ & $\bar{q}$ rotate with velocity $v = 1$ (UR quarks) $\Rightarrow v = \frac{r}{d/2} = \frac{2r}{d}$

\[ v_n \text{ distance from string element to rot. center} \]

\[ v_n \text{ velocity of string element} \]

\[ \Rightarrow M = \int \frac{dm}{\sqrt{1 - v_n^2}} = 2 \int_0^{d/2} \frac{\sigma \, dr}{\sqrt{1 - v_n^2}} = 2\sigma. \]

\[ \int_0^{d/2} \frac{dr}{\sqrt{1 - \left(\frac{2r}{d}\right)^2}} = 2\sigma \cdot \frac{d}{2} - \int_0^{1} \frac{d\xi}{\sqrt{1 - \xi^2}} = \frac{\pi}{2} \sigma \, d. \]

\[ (\text{arcsin} \xi) \]
The angular momentum (meson's spin) is

\[ J = \int \frac{r v \, dm}{\sqrt{1 - v^2}} = 2\sigma \int_0^{d_2} \frac{r v \, dr}{\sqrt{1 - v^2}} = \]

\[ = 2\sigma \int_0^{d_2} \frac{dr}{\sqrt{1 - (2r/d)^2}} = 2\sigma \left( \frac{d}{2} \right)^2 \int_0^{\pi/4} d\theta \frac{\sin^2 \theta}{\sqrt{1 - \sin^2 \theta}} = \]

\[ = \frac{\sigma d^2}{2} \cdot \frac{\pi}{4} \cdot \frac{\pi \sigma d^2}{8} \]

\[ \Rightarrow \text{meson mass} \quad M = \frac{\pi \sigma d^2}{2} \quad \text{Gasiorowicz} \]

\[ \Rightarrow \text{meson spin} \quad J = \frac{\pi \sigma d^2}{8} \quad \text{Rosner} \]

\[ \Rightarrow J = \frac{\pi}{8} \sigma \left( \frac{2M}{\pi \sigma} \right)^2 = \frac{1}{2\pi \sigma} M^2 \]

\[ \Rightarrow J = \frac{1}{2\pi \sigma} M^2 \quad \text{an example of a Regge trajectory} \]

In general, on the basis of phenomenological evidence, people noticed that

\[ J = J_0 + \lambda M^2 \quad \text{Chew & Frutisch} \]
$d_0$ = intercept
$\alpha' = \text{slope}$

Regge trajectory:

We get $\alpha' = \frac{1}{2 \pi \sigma}$

or $\sigma = \frac{1}{2 \pi \alpha'}$

Using $\sigma = 1 \text{ GeV/fm}$

$$\alpha' = \frac{1}{2 \pi \sigma} \approx \frac{5}{2 \pi} \text{ GeV}^{-2}$$

experimentally $\alpha' \approx 0.25 \text{ GeV}^{-2}$.

$\Rightarrow$ success of string approximation to strong interaction data led to proposal of string theory as the theory of strong interactions in the '60's.

$\Rightarrow$ that idea was killed by $e^+ e^- \rightarrow \mu^+ \mu^-$ (1973) & DIS data. String theory moved on to gravity '84.
Isospin symmetry: we had an isospin operator \( \vec{I} \) which was like angular momentum operator in 3d isospin fictitious space; as angular momentum it satisfied:

\[
[I_a, I_b] = i \varepsilon_{abc} I_c
\]

Compare with \( SU(2) \): group elements were \( e^{i \vec{J} \cdot \vec{a}} \) with \( \vec{J} = \frac{1}{2} \vec{a} \). We had the algebra for generators:

\[
[J_i, J_j] = i \varepsilon_{ijk} J_k
\]

Isospin symmetry is an \( SU(2) \) symmetry!

Fundamental representation \( \mathbf{1} \) of \( SU(2) \) is 2

\( \mathbf{2} \otimes \mathbf{2} = \mathbf{1} \oplus \mathbf{3} \)

2 \( \otimes \) 2 = 1 \( \oplus \) 3

\( ^{1} \text{(singlet)} \)

\( ^{3} \text{(triplet)} \)

Can have isospin 3 \( \mathbf{3} \) triplets:

\[
\begin{pmatrix}
\pi^+ \\
\pi^0 \\
\pi^-
\end{pmatrix},
\begin{pmatrix}
K^+ \\
K^0 \\
K^-
\end{pmatrix},
\begin{pmatrix}
\bar{\rho}^+ \\
\rho^0 \\
\bar{\rho}^-
\end{pmatrix},
\begin{pmatrix}
\bar{\pi}^+ \\
\pi^0 \\
\pi^-
\end{pmatrix},
\begin{pmatrix}
\bar{\rho}^+ \\
\rho^0 \\
\rho^-
\end{pmatrix},
\begin{pmatrix}
\bar{\pi}^+ \\
\pi^0 \\
\pi^-
\end{pmatrix}
\]

\( \ldots \)
Is this a symmetry of the Lagrangian that we wrote? Look at 2 flavors:

\[ q(x) = \begin{pmatrix} u(x) \\ d(x) \end{pmatrix}, \quad M = \begin{pmatrix} m_u & 0 \\ 0 & m_d \end{pmatrix} \Rightarrow \]

\[ \sum_{N^q = 2} 2 \text{ quarks} = \bar{q} \left( i \vec{\sigma} \cdot \vec{g} - m \right) q \]

First put \( m = 0 \Rightarrow L_0 = \bar{q}^f \cdot i \vec{\sigma} \cdot \vec{g} \Rightarrow \]

\[ SU(2) \text{ flavor transformation would be} \]

\[ e^{i \frac{\vec{\sigma} \cdot \vec{g}}{2}} \Rightarrow \bar{q} \rightarrow q' = e^{i \frac{\vec{\sigma} \cdot \vec{g}}{2}} \bar{q} \Rightarrow \text{\( x \)-indep.} \]

\[ \Rightarrow \bar{q} \rightarrow \bar{q}' = \bar{q} e^{-i \frac{\vec{\sigma} \cdot \vec{g}}{2}} \Rightarrow L_0 \text{ is invariant:} \]

\[ \bar{q} \cdot \vec{g} \Rightarrow q' \cdot \vec{g}' = \bar{q} e^{-i \frac{\vec{\sigma} \cdot \vec{g}}{2}} i \vec{g} e^{i \frac{\vec{\sigma} \cdot \vec{g}}{2}} q' = \bar{q} i \vec{g} \cdot \vec{g} \]

What about the mass term?

\[ \bar{q} m q \rightarrow \bar{q}' m q' = \bar{q} e^{-i \frac{\vec{\sigma} \cdot \vec{g}}{2}} m e^{i \frac{\vec{\sigma} \cdot \vec{g}}{2}} q \]

Write \( M = \begin{pmatrix} m_u + m_d & 0 \\ 0 & m_u - m_d \end{pmatrix} + \begin{pmatrix} m_u - m_d & 0 \\ 0 & m_u + m_d \end{pmatrix} \Rightarrow \]
\[ m = \frac{m_u + m_d}{2} + \frac{m_u - m_d}{2} \delta^3 \]

\[ \Rightarrow \quad q' = q \quad \text{with} \quad m_u + m_d = \frac{m_u + m_d}{2} + \frac{m_u - m_d}{2} \delta^3 \]

\[ \Rightarrow \quad m u = m d \Rightarrow \text{get exact } \text{SU}(2) \text{ flavor symmetry } (\text{global } \text{SU}(2) \text{ symmetry } \sim \tilde{Z} \text{ is independent of } \text{H}) \]

as \( m_u \neq m_d \) by a little bit \( \Rightarrow \text{SU}(2) \text{ flavor} \)

is (slightly) broken \( (\Rightarrow \text{hadron masses are different}) \)

\( \Rightarrow \) in reality the symmetry group is much larger!

\( \sim \text{SU}(2)_L \times \text{SU}(2)_R \) (more on this later.

(for massless quarks)

\( \Rightarrow \) Now, put the strange quark back in:

\[ q = \left( \begin{array}{c} u(x) \\ d(x) \\ s(x) \end{array} \right) \quad m = \left( \begin{array}{ccc} m_u & 0 & 0 \\ 0 & m_d & 0 \\ 0 & 0 & m_s \end{array} \right) \]

\( \Rightarrow \quad \mathbf{L}_3 \text{ quarks} = q \quad \left( i \gamma_5 \theta - m \right) q \quad \text{again.} \)

\( \Rightarrow \) one can check that if \( m_u = m_d = m_s \) then:

\( \mathbf{L} \) is invariant under \( \text{SU}(3) \text{ flavor} \) transform:

\[ q \rightarrow q' = e^{i \frac{\theta}{2}} q \quad \text{and} \quad T^a = \frac{1}{2} \lambda^a, \lambda^a \text{ Gell-Mann matrices} \]

\( a = 1, 2, \ldots, 8. \)
Now, let's look at mesons: $f$ = $g$ - states

$\Rightarrow 3 \otimes \bar{3} = 1 \oplus 8 \Rightarrow$ there should be a flavor octet and singlet:

- Pseudoscalar mesons: $\pi^+, \pi^0, \eta, K^+, K^0, \bar{K}^0, K^-$

form flavor - octet!

"The Eightfold Way!"

$\bar{f}'$ = flavor singlet!

$v = (v_u + v_d + v_s)\frac{1}{2}$.

Vector mesons = the same story!

What about baryons? $g$ $g$ $g$ - states $\Rightarrow$

$3 \otimes 3 \otimes 3 = 1 \oplus 8 \oplus 8 \oplus 10$

$\frac{1}{2}$ baryons: $\rho, \eta, \Sigma^+, \Sigma^0, \Sigma^-, \Xi^+, \Xi^0, \Xi^-$ form an octet!

Baryon decuplet $\Rightarrow$ that is the 10!}

$\Rightarrow$ as $SU(3)$ flavor is not an exact symmetry, all masses are different - broken symmetry!