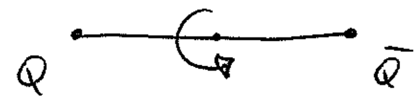
Last time | Used the string model of strong interactions to derive $V(r) = \sigma r$ at large r.

Spinning string:



$$\boxed{J = \frac{1}{2\pi\sigma} M^2}$$

$\Rightarrow$ Regge trajectory

~ works for hadron masses

J: spin (angular momentum)

M: mass (energy)

Quark Symmetries Revisited (cont'd)

$$\boxed{\mathcal{L}_{quarks} = \bar{q}\,(i\not{\partial} - m)\,q}$$

for $N_f = 2 \Rightarrow$ $m = \begin{pmatrix} m_u & 0 \\ 0 & m_d \end{pmatrix}$, $q(x) = \begin{pmatrix} u(x) \\ d(x) \end{pmatrix}$

$$q(x) \to q'(x) = e^{i\vec{\alpha}\cdot\frac{\vec{\sigma}}{2}}\, q(x) \quad \sim \text{global } SU(2)$$

This is an exact symmetry of $\mathcal{L}_{quarks}$ <u>only</u> if $m_u = m_d$. Otherwise, for $m_u \neq m_d$ it is broken explicitly.

$$\Rightarrow m = \frac{m_u+m_d}{2}\mathbb{1} + \frac{m_u-m_d}{2}\sigma^3$$

$$\Rightarrow \bar{q}' m q' = \bar{q}\frac{m_u+m_d}{2} q + \frac{m_u-m_d}{2}\bar{q}\underbrace{e^{-i\vec{\alpha}\cdot\frac{\vec{\sigma}}{2}}\sigma^3 e^{i\vec{\alpha}\cdot\frac{\vec{\sigma}}{2}}}_{\neq \sigma^3} q$$

$\Rightarrow$ if $m_u = m_d$ $\Rightarrow$ get exact $SU(2)$ flavor symmetry (global $SU(2)$ symmetry ~ $\vec{\alpha}$ is independent of x^μ)

as $m_u \neq m_d$ by a little bit $\Rightarrow$ $SU(2)$ flavor is (slightly) broken. ($\Rightarrow$ hadron masses are different)

$\Rightarrow$ in reality the symmetry group is much larger!

~ $SU(2)_R \times SU(2)_L$ ~ more on this later.
(for massless quarks)

$\Rightarrow$ Now, put the strange quark back in:

$$q = \begin{pmatrix} u(x) \\ d(x) \\ s(x) \end{pmatrix}, \quad m = \begin{pmatrix} m_u & 0 & 0 \\ 0 & m_d & 0 \\ 0 & 0 & m_s \end{pmatrix}$$

$$\Rightarrow \mathcal{L}^{N_f=3}_{\text{quarks}} = \bar{q}\,(i\gamma\cdot\partial - m)\,q \quad \text{again.}$$

$\Rightarrow$ one can check that if $m_u = m_d = m_s$ then $\mathcal{L}$ is invariant under $SU(3)$ flavor transform:

$$q \to q' = e^{i\vec{\alpha}\cdot\vec{T}} q, \quad T^a = \frac{1}{2}\lambda^a, \quad \lambda^a \sim \text{Gell-Mann matrices}$$

$a = 1, 2, \ldots, 8.$

⇒ as $m_u \neq m_d \neq m_s$ SU(3) is not an exact flavor symmetry.

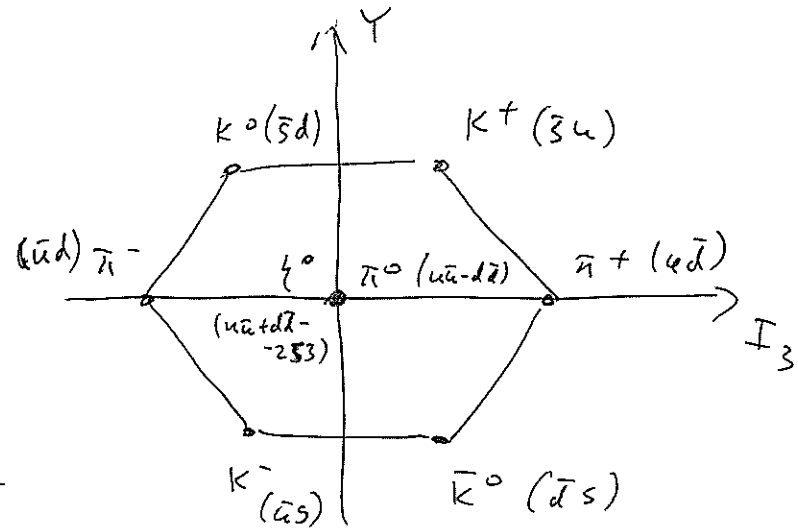
Now, let's look at mesons: $\bar{q}q$ ~ states

⇒ $3 \otimes \bar{3} = 1 \oplus 8$ ⇒ there should be a flavor octet and singlet:

pseudoscalar mesons $\pi^+, \pi^-, \pi^0, \eta^0, K^+, K^0, \bar{K}^0, K^-$ form flavor-octet!
"The Eightfold Way"!

η' ~ flavor singlet! $\sim (\bar{u}u + \bar{d}d + \bar{s}s)\frac{1}{\sqrt{3}}$.

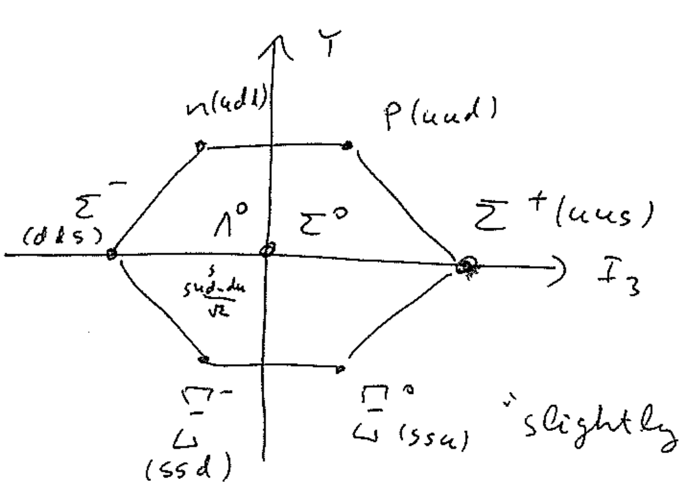
vector mesons ~ the same story!



What about baryons? qqq-states ⇒

$3 \otimes 3 \otimes 3 = 1 \oplus 8 \oplus 8 \oplus 10$

$\frac{1}{2}^+$ baryons: $p, n, \Sigma^+, \Sigma^-, \Sigma^0, \Lambda^0, \Xi^0, \Xi^-$ ~ form an octet!



baryon decuplet ~ that's the 10!

⇒ as SU(3) flavor is not exact, all masses are "slightly" different ~ broken symmetry!

$(m_{\Xi^0} = 1315$ MeV, $m_p = 938$ MeV$)$

Gell-Mann – Okubo Mass Formula

$\Rightarrow$ Note that $m_p \neq 2m_u + m_d$ $\Rightarrow$ most of the mass is due to gluonic interactions $\Rightarrow$

$\Rightarrow$ write $m_p = m_0 + 2m_u + m_d \simeq m_0 + 3m_u$ ($m_d \simeq m_u$)

$\Sigma^+ = uus$ $\qquad m_\Sigma = m_0 + 2m_u + m_s$

$\Xi^0 = uss$ $\qquad m_\Xi = m_0 + m_u + 2m_s$ $\qquad m_\Lambda = m_\Sigma$

$\Lambda^0 = uds$

$\qquad m_\Lambda = m_0 + 2m_u + m_s$

$$\Rightarrow \frac{m_\Sigma + 3m_\Lambda}{2} = m_p + m_\Xi$$ for $\frac{1}{2}^+$ baryon octet.

$m_p = 938\,\text{MeV}$, $m_\Lambda = 1116\,\text{MeV}$, $m_\Xi = 1315\,\text{MeV}$, $m_\Sigma = 1189\,\text{MeV}$

$LHS = 2268.5\,\text{MeV}$, $RHS = 2253\,\text{MeV}$ ~ close enough!

For $\frac{3}{2}^+$ baryon decuplet get

$\Omega^- = sss$

$\Xi^{*-} = ssd$

$\Sigma^{*+} = suu$

$\Delta^{++} = uuu$

$$m_\Omega - m_{\Xi^*} = m_{\Xi^*} - m_{\Sigma^*} = m_{\Sigma^*} - m_\Delta$$

~ also works

~ was used to <u>predict</u> the mass of Ω^--baryon.

Flavor SU(2) and SU(3) Symmetries.

Let's go back to 2-flavor QCD:

$$\mathcal{L}^{N_f=2}_{quarks} = \bar{q}\,(i\gamma\cdot\partial - m)\,q \quad , \quad m = \begin{pmatrix} m_u & 0 \\ 0 & m_d \end{pmatrix}$$

We saw that if $m_u = m_d$ we have SU(2) flavor symmetry in the lagrangian.

$\Rightarrow$ However, masses of hadrons are much larger than current quark masses ($m_p \gg 2m_u + m_d$).

$\Rightarrow$ the flavor symmetry is more due to the fact that quark masses are small!

$\Rightarrow$ put $m_u = m_d = 0$

$$\Rightarrow \mathcal{L} = \bar{q}\; i\gamma\cdot\partial\, q .$$

Write $q = q_L + q_R = \underbrace{\frac{1-\gamma_5}{2}\,q}_{q_L} + \underbrace{\frac{\mathbb{1}+\gamma_5}{2}\,q}_{q_R}$

$$\gamma_5 = i\gamma^0\gamma^1\gamma^2\gamma^3 \quad , \quad \{\gamma_5, \gamma^\mu\} = 0 \quad , \quad \gamma_5^{\dagger} = \gamma_5$$

$$\gamma_5 = \begin{pmatrix} 0 & \mathbb{1} \\ 1 & 0 \end{pmatrix} \quad , \quad \gamma_5\gamma_5 = \mathbb{1}$$

Projection operators $P_L = \frac{\mathbb{1}-\gamma_5}{2}$, $P_R = \frac{\mathbb{1}+\gamma_5}{2}$

$$\Rightarrow P_L^2 = \left(\frac{\mathbb{1}-\gamma_5}{2}\right)^2 = \frac{\mathbb{1} - 2\gamma_5 + \overbrace{\gamma_5^2}^{=\mathbb{1}}}{4} = P_L$$

$P_R^2 = P_R$, $\quad P_R P_L = \frac{1\!\!1+\gamma_5}{2}\,\frac{1\!\!1-\gamma_5}{2} = \frac{1\!\!1-\gamma_5^2}{4} = 0.$

For massless particles they project on different helicity states. $P_L = \frac{1}{2}\begin{pmatrix} 1\!\!1 & -1\!\!1 \\ -1\!\!1 & 1\!\!1 \end{pmatrix}$, $\quad P_R = \begin{pmatrix} 1\!\!1 & 1\!\!1 \\ 1\!\!1 & 1\!\!1 \end{pmatrix}$.

Now, $\bar{q} = q^+\gamma^0 \Rightarrow \quad q^+ = q^+\left(\frac{1\!\!1-\gamma_5}{2}\right) + q^+\left(\frac{1\!\!1+\gamma_5}{2}\right) = q_L^+ + q_R^+$

$$\Rightarrow \bar{q} = \underbrace{\bar{q}\,\frac{1+\gamma_5}{2}}_{\bar{q}_L} + \underbrace{\bar{q}\,\frac{1\!\!1-\gamma_5}{2}}_{\bar{q}_R} \qquad \text{as } \{\gamma_5, \gamma_0\} = 0.$$

$$\Rightarrow \mathcal{L} = \left[\bar{q}\,\frac{1\!\!1+\gamma_5}{2} + \bar{q}\,\frac{1\!\!1-\gamma_5}{2}\right] i\gamma\cdot\partial \left[\frac{1\!\!1-\gamma_5}{2}q + \frac{1\!\!1+\gamma_5}{2}q\right]$$

survives

survives

$$\Rightarrow \boxed{\mathcal{L} = \bar{q}_L\, i\gamma\cdot\partial\, q_L + \bar{q}_R\, i\gamma\cdot\partial\, q_R}$$

Now, this lagrangian is <u>separately</u> invariant under $q_L \to e^{i\vec{\alpha}_L\cdot\frac{\vec{\sigma}}{2}} q_L$ and $q_R \to e^{i\vec{\alpha}_R\frac{\vec{\sigma}}{2}} q_R$

$\Rightarrow$ the net symmetry is $\boxed{SU(2)_L \otimes SU(2)_R}$. Chiral Symmetry

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$\Rightarrow$ Now add back the mass term with $m_u = m_d$:

$$-m\bar{q}q = -m\left[\bar{q}\,\frac{1\!\!1+\gamma_5}{2} + \bar{q}\,\frac{1\!\!1-\gamma_5}{2}\right]\left[\frac{1\!\!1-\gamma_5}{2}q + \frac{1\!\!1+\gamma_5}{2}q\right]$$

$= -m\left[\bar{q}_L q_R + \bar{q}_R q_L\right] \Rightarrow$ mixing $\Rightarrow$ need $\vec{\alpha}_R = \vec{\alpha}_L$

What we know so far: for $N_f = 2$

$$\boxed{SU(2)_L \otimes SU(2)_R}$$

$$\Big\downarrow \; m_u = m_d \neq 0$$

$$\boxed{SU(2)}$$

$$\Big\downarrow \; m_u \neq m_d \neq 0$$

Nothing

similarly for $N_f = 3$:

$$\boxed{SU(3)_L \otimes SU(3)_R}$$

$$\Big\downarrow \; m_u = m_d = m_s \neq 0$$

$$\boxed{SU(3)}$$

$$\Big\downarrow \; m_u \neq m_d \neq m_s \neq 0$$

Nothing

$\Rightarrow$ $SU(2)_L \otimes SU(2)_R$ is broken down to $SU(2)$.

What are the conserved currents of $SU(2)_R \otimes SU(2)_L$?

Noether theorem: every continuous symmetry gives a conservation law!

Go back to the massless case:

$$\mathcal{L} = \bar{q}_L \, i\gamma\cdot\partial \, q_L + \bar{q}_R \, i\gamma\cdot\partial \, q_R$$

$$q_L \underset{SU(2)}{\rightarrow} e^{i\vec{\alpha}\cdot\frac{\vec{\sigma}}{2}} q_L \Rightarrow \text{if } \vec{\alpha} \text{ is small } q_L \rightarrow \left(1 + i\delta\vec{\alpha}\cdot\frac{\vec{\sigma}}{2}\right) q_L$$
$$= q_L + \delta q_L$$

$\Rightarrow \delta\mathcal{L} = 0$ as it is a symmetry $\Rightarrow$

$$0 = \delta\mathcal{L} = \frac{\delta\mathcal{L}}{\delta q_L}\delta q_L + \frac{\delta\mathcal{L}}{\delta\bar{q}_L}\delta\bar{q}_L + \frac{\delta\mathcal{L}}{\delta(\partial_\mu q_L)}\delta(\partial_\mu q_L) +$$

$$+ \delta(\partial_\mu\bar{q}_L)\frac{\delta\mathcal{L}}{\delta(\partial_\mu\bar{q}_L)} = \underbrace{\left[\frac{\delta\mathcal{L}}{\delta q_L} - \partial_\mu\frac{\delta\mathcal{L}}{\delta(\partial_\mu q_L)}\right]}_{=0\ (EOM)}\delta q_L + \partial_\mu\left(\frac{\delta\mathcal{L}}{\delta(\partial_\mu q_L)}\delta q_L\right)$$

$$+ \delta\bar{q}_L\underbrace{\left[\frac{\delta\mathcal{L}}{\delta\bar{q}_L} - \partial_\mu\frac{\delta\mathcal{L}}{\delta(\partial_\mu\bar{q}_L)}\right]}_{=0\ (EOM)} + \partial_\mu\left[\delta\bar{q}_L\cancel{\frac{\delta\mathcal{L}}{\delta(\partial_\mu\bar{q}_L)}}\right] \rightarrow 0 \text{ for our } \mathcal{L}$$

$$\Rightarrow 0 = \partial_\mu\left(\frac{\delta\mathcal{L}}{\delta(\partial_\mu q_L)}\delta q_L\right)$$

$$\Rightarrow 0 = \partial_\mu\left[\bar{q}_L \, i\gamma^\mu \, \delta q_L\right] = \partial_\mu\left[\bar{q}_L \, i\gamma^\mu \, i\frac{\delta\vec{\alpha}\cdot\vec{\sigma}}{2} q_L\right]$$

Def. $\partial_\mu j^{i\mu}_L = 0$ where $\boxed{j^{i\mu}_L = \bar{q}_L \gamma^\mu \frac{\sigma^i}{2} q_L}$

left-handed isospin current.

Def. Similarly define $\boxed{j^{i\mu}_R = \bar{q}_R \gamma^\mu \frac{\sigma^i}{2} q_R}$ ~ right-handed isospin current.

$\partial_\mu j^{i\mu}_R = 0$

Alternatively can define

Def.

$$j^i_\mu = j^i_{L\mu} + j^i_{R\mu} = \bar{q} \gamma_\mu \frac{\sigma^i}{2} q \quad \sim \text{vector isospin current}$$

$$j^i_{5\mu} = j^i_{R\mu} - j^i_{L\mu} = \bar{q} \gamma_\mu \gamma_5 \frac{\sigma^i}{2} q \quad \sim \text{axial vector isospin current}$$

<u>Define</u> charges: $\boxed{Q^i_{L,R}(t) = \int d^3x \; j^i_{L,R\,0}(\vec{x}, t)}$

$$\frac{dQ^i_L(t)}{dt} = \int d^3x \frac{\partial j^i_{L0}(\vec{x},t)}{\partial t} = \int d^3x \Big[\underbrace{\partial_\mu j^{i\mu}_L}_{=0 \text{ (conserved current)}} - \vec{\nabla}\cdot\vec{j}^{\,i}_L\Big]$$

$$= -\int d^3x \; \vec{\nabla}\cdot\vec{j}^{\,i}_L = 0$$

surface term

$\Rightarrow$ charges are conserved!

$\Rightarrow$ the charges are <u>generators</u> of $SU(2)_L \otimes SU(2)_R$!

One can show that they form the chiral $SU(2)_L \otimes SU(2)_R$ algebra:

$$\boxed{\begin{aligned} [Q^i_L, Q^j_L] &= i\,\varepsilon_{ijk}\, Q^k_L \quad &\leftarrow SU(2)_L \\ [Q^i_R, Q^j_R] &= i\,\varepsilon_{ijk}\, Q^k_R \quad &\leftarrow SU(2)_R \\ [Q^i_L, Q^j_R] &= 0 \quad &\sim \text{commute with each other.} \end{aligned}}$$

Let's show how Q_L^i generate $SU(2)_L$ transformations.

Let's calculate $[Q_L^i(t), q_L(t,\vec{x})]$:

$$[Q_L^i(t), \underset{\uparrow \text{ spinor index } 1,2,3,4}{q_{L\,\alpha}^{\overset{\text{flavor index } 1,2}{a}}}(\vec{x},t)] = \int d^3x' \left[\bar{q}_L \gamma_0 \frac{\sigma^i}{2} q_L(\vec{x}',t), \right.$$

$$\left. q_{L\,\alpha}^{a}(\vec{x},t)\right] = \int d^3x' \left[\underbrace{\bar{q}_{L\,\beta}^{\,b}(\vec{x}',t)\,(\gamma^0)_{\beta\delta}}_{q_{L\,\delta}^{+\,b}(\vec{x}',t) \,=\, q^{+\,b}_{\delta'}\left(\frac{1-\gamma_5}{2}\right)_{\delta'\delta}} \left(\frac{\sigma^i}{2}\right)_{bc} \underbrace{q_{L\,\delta}^{c}(\vec{x}',t)}_{=\left(\frac{1-\gamma_5}{2}\right)_{\delta\delta''} q^c_{\delta''}}, \right.$$

$$\left. q_{L\,\alpha}^{a}(\vec{x},t)\right] =$$

$$= \int d^3x' \cdot \left(\frac{1-\gamma_5}{2}\right)_{\delta'\delta} \left(\frac{1-\gamma_5}{2}\right)_{\delta\delta''} \left(\frac{1-\gamma_5}{2}\right)_{\alpha\alpha'} \cdot \left(\frac{\sigma^i}{2}\right)_{bc} \cdot$$

$$\cdot \left[q^{+\,b}_{\delta'}(\vec{x}',t)\, q^c_{\delta''}(\vec{x}',t),\; q^a_{\alpha'}(\vec{x},t)\right]$$

$\Rightarrow$ use the anti-commutation relations

$$\{ q^a_\alpha(\vec{x},t),\, q^{+\,b}_\beta(\vec{x}',t)\} = \delta^{ab}\delta_{\alpha\beta}\,\delta(\vec{x}-\vec{x}').$$

$$[Q_L^i(t), q_{L\,\alpha}^{a}(\vec{x},t)] = \left(\frac{1-\gamma_5}{2}\right)_{\delta'\delta''} \left(\frac{1-\gamma_5}{2}\right)_{\alpha\alpha'} \left(\frac{\sigma^i}{2}\right)_{bc} \cdot \int d^3x' \cdot$$

$$\cdot(-)\left\{ q^a_{\alpha'}(\vec{x},t),\, q^{+\,b}_{\delta'}(\vec{x}',t)\right\} q^c_{\delta''}(\vec{x}',t) = -\left(\frac{1-\gamma_5}{2}\right)_{\delta'\delta''} \cdot$$

$$\cdot\left(\frac{1-\gamma_5}{2}\right)_{\alpha\alpha'} \left(\frac{\sigma^i}{2}\right)_{bc} \delta^{ab}\,\delta_{\alpha'\delta'}\, q^c_{\delta''}(\vec{x},t) = -\left(\frac{\sigma^i}{2}\right)_{ac} \cdot$$

$$\cdot\left(\frac{1-\gamma_5}{2}\right)_{\alpha\delta''} q^c_{\delta''}(\vec{x},t) = -\left(\frac{\sigma^i}{2}\right)_{ac} q^c_{L\,\alpha}(\vec{x},t)$$

$\Rightarrow$ get $\left[Q_L^i(t), q_L(\vec{x},t) \right] = -\frac{\sigma^i}{2} q_L(\vec{x},t)$

$\Rightarrow$ can show that

$$e^{-i\vec{\alpha}_L\cdot\vec{Q}_L(t)} q_L(\vec{x},t)\, e^{i\vec{\alpha}_L\cdot\vec{Q}_L(t)} = e^{i\vec{\alpha}_L\cdot\frac{\vec{\sigma}}{2}} q_L(\vec{x},t)$$

$\Rightarrow$ Q_L's generate transformations of $SU(2)_L$

$\Rightarrow$ Q_R's —"— of $SU(2)_R$ (can show similarly).

c.f. $\hat{O}(t) = e^{i\hat{H}t}\hat{O}(0)e^{-i\hat{H}t} = e^{t\frac{\partial}{\partial t'}}\hat{O}(t')\big|_{t'=0}$ $\Rightarrow$ $\hat{H}$ generates time translations

bring back the <u>strange quark</u> $\Rightarrow$ now can perform the same decomposition and for $m_u = m_d = m_s = 0$ have $SU(3)_R \otimes SU(3)_L$ chiral symmetry.

$$\mathcal{L} = \bar{q}_L i\gamma\cdot\partial q_L + \bar{q}_R i\gamma\cdot\partial q_R$$

$\Rightarrow$ invariant under $q_L \to e^{i\vec{\alpha}_L\cdot\vec{T}} q_L$, $q_R \to e^{i\vec{\alpha}_R\cdot\vec{T}} q_R$

$T^a = \frac{\lambda^a}{2}$ ~ generators of $SU(3)$.

<u>Problem</u>: $SU(3)_L \otimes SU(3)_R$ would imply twice as many degenerate multiplets of hadrons: 8 0^- mesons should come in together with 8 0^+ mesons, etc.

$\Rightarrow$ This does not happen in nature. Why?