Last time: **Flavor SU(2) and SU(3) Symmetries** (cont'd)

$$\boxed{\mathcal{L}^{N_f=2}_{\text{quarks}} = \bar{q}\left[i\not{\partial} - M\right] q}, \quad q = \begin{pmatrix} u \\ d \end{pmatrix}, \quad m = \begin{pmatrix} m_u & 0 \\ 0 & m_d \end{pmatrix}$$

The Lagrangian is $SU(2)_L \otimes SU(2)_R$ invariant if $m_u = m_d = 0$. In real life $m_u \neq m_d \neq 0$:

$$\boxed{SU(2)_L \otimes SU(2)_R} \xrightarrow{m_u = m_d \neq 0} \boxed{SU(2)} \xrightarrow{m_u \neq m_d \neq 0} \text{Nothing}$$

(ditto for $SU(3)$)

Def. Conserved currents:

$$\boxed{j_\mu^i = \bar{q}\gamma_\mu \frac{\sigma^i}{2} q}, \qquad \boxed{j_{\mu 5}^i = \bar{q}\gamma_\mu\gamma_5 \frac{\sigma^i}{2} q}$$

vector isospin current, axial vector isospin current

or $j_{L\mu}^i = \bar{q}_L \gamma_\mu \frac{\sigma^i}{2} q_L$, $j_{R\mu}^i = \bar{q}_R \gamma_\mu \frac{\sigma^i}{2} q_R$

Def. Charges: $\boxed{Q_{L,R}^i(t) \equiv \int d^3x \, j_{L,R}^{i0}(\vec{x}, t)}$.

$$\boxed{\begin{aligned} [Q_L^i, Q_L^j] &= i\varepsilon^{ijk} Q_L^k \\ [Q_R^i, Q_R^j] &= i\varepsilon^{ijk} Q_R^k \\ [Q_L^i, Q_R^j] &= 0 \end{aligned}}$$

$\leftarrow SU(2)_L$

$\leftarrow SU(2)_R$

Proved that $[Q_L^i(t), q_L(\vec{x}, t)] = -\frac{\sigma^i}{2} q_L(\vec{x}, t)$

$\Rightarrow$ Q_L ~ generates $SU(2)$ transformations

Let's show how Q_L^i generate $SU(2)_L$ transformations. (72)

Let's calculate $[\,Q_L^i(t),\ q_L(t,\vec{x})\,]$:

$$[\,Q_L^i(t),\ q_{L\,\alpha}^{\ a}(\vec{x},t)\,] = \int d^3x' \,[\, \bar{q}_L \gamma_0 \frac{\sigma^i}{2} q_L(\vec{x}',t),$$

(a: flavor index 1,2; α: spinor index 1,2,3,4)

$$q_{L\,\alpha}^{\ a}(\vec{x},t)\,] = \int d^3x' \,[\, \underbrace{\bar{q}_{L\,\beta}^{\ b}(\vec{x}',t)\,(\gamma^0)_{\beta\delta}}_{q_{L\,\delta}^{+\,b}(\vec{x}',t)\ =\ q^{+\,b}_{\delta'}\left(\frac{1-\gamma_5}{2}\right)_{\delta'\delta}} \left(\frac{\sigma^i}{2}\right)_{bc} \underbrace{q_{L\,\delta}^{\ c}(\vec{x}',t)}_{=\left(\frac{1-\gamma_5}{2}\right)_{\delta\delta''} q^c_{\delta''}},$$

$$q_{L\,\alpha}^{\ a}(\vec{x},t)\,] =$$

$$= \int d^3x' \cdot \left(\frac{1-\gamma_5}{2}\right)_{\delta'\delta} \left(\frac{1-\gamma_5}{2}\right)_{\delta\delta''} \left(\frac{1-\gamma_5}{2}\right)_{\alpha\alpha'} \cdot \left(\frac{\sigma^i}{2}\right)_{bc} \cdot$$

$$[\, q^{+\,b}_{\delta'}(\vec{x}',t)\, q^{c}_{\delta''}(\vec{x}',t),\ q^{a}_{\alpha'}(\vec{x},t)\,]$$

$\Rightarrow$ use the anti-commutation relations

$$\{\, q^a_\alpha(\vec{x},t),\ q^{+\,b}_\beta(\vec{x}',t)\,\} = \delta^{ab}\delta_{\alpha\beta}\,\delta(\vec{x}-\vec{x}').$$

$$[\,Q_L^i(t),\ q_{L\,\alpha}^{\ a}(\vec{x},t)\,] = \left(\frac{1-\gamma_5}{2}\right)_{\delta'\delta''} \left(\frac{1-\gamma_5}{2}\right)_{\alpha\alpha'} \left(\frac{\sigma^i}{2}\right)_{bc} \cdot \int d^3x' \cdot$$

$$\cdot(-)\,\{\, q^a_{\alpha'}(\vec{x},t),\ q^{+\,b}_{\delta'}(\vec{x}',t)\,\}\, q^c_{\delta''}(\vec{x}',t) = -\left(\frac{1-\gamma_5}{2}\right)_{\delta'\delta''} \cdot$$

$$\cdot\left(\frac{1-\gamma_5}{2}\right)_{\alpha\alpha'} \left(\frac{\sigma^i}{2}\right)_{bc} \delta^{ab}\,\delta_{\alpha'\delta'}\, q^c_{\delta''}(\vec{x},t) = -\left(\frac{\sigma^i}{2}\right)_{ac} \cdot$$

$$\cdot\left(\frac{1-\gamma_5}{2}\right)_{\alpha\delta''} q^c_{\delta''}(\vec{x},t) = -\left(\frac{\sigma^i}{2}\right)_{ac} q^c_{L\,\alpha}(\vec{x},t)$$

$\Rightarrow$ get $[\,Q_L^{\,i}(t),\ q_L(\vec{x},t)\,] = -\frac{\sigma^i}{2}\, q_L(\vec{x},t)$

$\Rightarrow$ can show that

$$e^{-i\vec{\alpha}_L\cdot\vec{Q}_L(t)}\ q_L(\vec{x},t)\ e^{i\vec{\alpha}_L\cdot\vec{Q}_L(t)} = e^{i\vec{\alpha}_L\cdot\frac{\vec{\sigma}}{2}}\ q_L(\vec{x},t)$$

$\Rightarrow$ Q_L's generate transformations of $SU(2)_L$

$\Rightarrow$ Q_R's —"— of $SU(2)_R$ (can show similarly).

c.f. $\hat{O}(t) = e^{i\hat{H}t}\,\hat{O}(0)\,e^{-i\hat{H}t} = e^{t\frac{\partial}{\partial t'}}\,\hat{O}(t')\big|_{t'=0}$ $\Rightarrow$ $\hat{H}$ generates time translations

bring back the strange quark $\Rightarrow$ now can perform the same decomposition and for $m_u = m_d = m_s = 0$ have $SU(3)_R \otimes SU(3)_L$ chiral symmetry.

$$\mathcal{L} = \bar{q}_L\, i\gamma\cdot\partial\, q_L + \bar{q}_R\, i\gamma\cdot\partial\, q_R$$

$\Rightarrow$ invariant under $q_L \to e^{i\vec{\alpha}_L\cdot\vec{T}} q_L$, $q_R \to e^{i\vec{\alpha}_R\cdot\vec{T}} q_R$

$T^a = \frac{\lambda^a}{2}$ ~ generators of $SU(3)$.

Problem: $SU(3)_L \otimes SU(3)_R$ would imply twice as many degenerate multiplets of hadrons: 8 0^- mesons should come in together with 8 0^+ mesons, etc.

$\Rightarrow$ this does not happen in nature. Why?

$\Rightarrow$ you may say: well, as $m_u, m_d, m_s \neq 0$

$\Rightarrow$ $SU(3)_R \otimes SU(3)_L$ is broken.

But then one would not have any multiplets at all, one would not have the Eight fold Way, etc...

$\Rightarrow$ OK, you would say, we have $SU(3)$ flavor if $m_u = m_d = m_s \neq 0$.

$\Rightarrow$ But as $m_u \neq m_d \neq m_s$ $SU(3)$ flavor is just as broken as $SU(3)_L \otimes SU(3)_R$.

NB Fact of the matter is both $SU(3)_L \otimes SU(3)_R$ and $SU(3)$ are broken "slightly". The real symmetry breaking is $SU(3)_L \otimes SU(3)_R \to SU(3)$ is done through <u>spontaneous symmetry breaking</u>. (SSB)

SSB has <u>nothing</u> to do with quark masses!

$\Rightarrow$ SSB would also explain why the masses of hadrons are so much higher than the current masses of quarks they are made of.

($m_p = 938$ MeV , $2m_u + m_d < 30$ MeV)

$$\frac{2m_u + m_d}{m_p} \simeq 3\%.$$

$SU(3)_L \otimes SU(3)_R$

↓ $m_u = m_d = m_s \neq 0$

$SU(3)$

↓ $m_u \neq m_d \neq m_s \neq 0$

Nothing

equally broken

$\Rightarrow$ $SU(3)$ works (the Eight fold way) approximately

$\Rightarrow$ $SU(3)_L \otimes SU(3)_R$ <u>does not work!</u> (no 0^+ meson octet, no $(\frac{1}{2})^-$ baryon octet, ...)

Solution:

$SU(3)_L \otimes SU(3)_R$

↓ $m_u = m_d = m_s \neq 0$ + **SSB** (Spontaneous (chiral) Symmetry Breaking)

$SU(3)$

↓ $m_u \neq m_d \neq m_s \neq 0$

Nothing

SSB makes this breaking much worse than that

Expand $\mathcal{L}$ near the vacuum: $\sigma = v + \sigma'$

$$\Rightarrow \mathcal{L} = \frac{1}{2}\partial_\mu\sigma'\partial^\mu\sigma' + \frac{1}{2}\partial_\mu\vec{\pi}\,\partial^\mu\vec{\pi} + \frac{\mu^2}{2}\left[(v+\sigma')^2+\vec{\pi}^2\right]$$

$$-\frac{\lambda}{4}\left[(v+\sigma')^2+\vec{\pi}^2\right]^2 = \frac{1}{2}\partial_\mu\sigma'\partial_\mu\sigma' + \frac{1}{2}\partial_\mu\vec{\pi}\,\partial^\mu\vec{\pi} + \underbrace{\text{const}}_{\text{drop}}$$

$$+\sigma'\underbrace{\left[\mu^2 v - \frac{\lambda}{4}\cdot 4v^3\right]}_{\to 0} + \sigma'^2\left[\frac{\mu^2}{2} - \frac{\lambda}{4}\cdot\left(2v^2+4v^2\right)\right] +$$

$$+\vec{\pi}^2\underbrace{\left[\frac{\mu^2}{2} - \frac{\lambda}{4}\cdot 2v^2\right]}_{\to 0} - \frac{\lambda}{4}\left[4\sigma' v(\sigma'^2+\vec{\pi}^2) + (\sigma'^2+\vec{\pi}^2)^2\right]$$

$$= \frac{1}{2}\partial_\mu\sigma'\partial_\mu\sigma' + \frac{1}{2}\partial_\mu\vec{\pi}\,\partial^\mu\vec{\pi} - \mu^2\sigma'^2 - \lambda v\,\sigma'(\sigma'^2+\vec{\pi}^2)$$

$$-\frac{\lambda}{4}\left(\sigma'^2+\vec{\pi}^2\right)^2.$$

$\Rightarrow$ now $\vec{\pi}$'s have no $\vec{\pi}^2$ term $\Rightarrow$ $\vec{\pi}$ field is <u>massless</u> in agreement with Goldsone th'm!

Non-Abelian σ-Model

Let's illustrate how the chiral $SU(3)_L \otimes SU(3)_R$ symmetry is broken in QCD. As an example consider breaking of $SU(2)_L \otimes SU(2)_R$ symmetry. Start with the Lagrangian:

$$\mathcal{L} = \frac{1}{2}\left(\partial_\mu\sigma\,\partial^\mu\sigma + \partial_\mu\vec{\pi}\,\partial^\mu\vec{\pi}\right) + \frac{\mu^2}{2}\left(\sigma^2+\vec{\pi}^2\right) - \frac{\lambda}{4}\left[\sigma^2+\vec{\pi}^2\right]^2$$

$\mu^2, \lambda > 0$, $\overset{\text{isoscalar}}{\sigma}$, $\vec{\pi} = \underbrace{(\pi_1, \pi_2, \pi_3)}_{\text{iso triplet pions}}$ ~ real fields

Define a 2×2 matrix field $\Sigma = \sigma \mathbb{1} + i\,\vec{\tau}\cdot\vec{\pi}$

τ^1, τ^2, τ^3 = Pauli matrices (we use τ to not confuse them with σ)

$$\Rightarrow \mathrm{tr}\left[\Sigma \quad \Sigma^+\right] = \mathrm{tr}\left[\sigma^2 \mathbb{1} + i\,\vec{\tau}\cdot\vec{\pi}(-i)\,\vec{\tau}\cdot\vec{\pi}\right]$$

$$= 2\sigma^2 + 2\vec{\pi}^2 \qquad \text{as } \mathrm{tr}\,\tau^i\tau^j = 2\delta^{ij}$$

$$\Rightarrow \mathrm{tr}\left[\partial_\mu \Sigma\ \partial^\mu \Sigma^+\right] = 2\left[\partial_\mu\sigma\,\partial^\mu\sigma + \partial_\mu\vec{\pi}\ \partial^\mu\vec{\pi}\right]$$

$$\Rightarrow \mathcal{L}_\Sigma = \frac{1}{4}\left[\mathrm{tr}\ \partial_\mu\Sigma\ \partial^\mu\Sigma^+\right] + \frac{\mu^2}{4}\mathrm{tr}\left[\Sigma\Sigma^+\right] - \frac{\lambda}{16}\left(\mathrm{tr}\left[\Sigma\Sigma^+\right]\right)^2$$

Now add "quarks": (originaly they were protons and neutrons): $q = \begin{pmatrix} u \\ d \end{pmatrix}$ or $\begin{pmatrix} p \\ n \end{pmatrix} = q^N$

$$\mathcal{L} = \bar{q}^N i\gamma\cdot\partial q^N - g\,\bar{q}^N\left[\sigma\mathbb{1} + i\,\vec{\tau}\cdot\vec{\pi}\,\gamma_5\right]q^N + \mathcal{L}_\Sigma$$

Such that

$$\mathcal{L} = \bar{q}^N i\gamma\cdot\partial q^N - g\,\bar{q}^N\left[\sigma\mathbb{1} + i\,\vec{\tau}\cdot\vec{\pi}\,\gamma_5\right]q^N +$$
$$+ \frac{1}{2}\left(\partial_\mu\sigma\,\partial^\mu\sigma + \partial_\mu\vec{\pi}\ \partial^\mu\vec{\pi}\right) + \frac{\mu^2}{2}\left(\sigma^2 + \vec{\pi}^2\right) - \frac{\lambda}{4}\left(\sigma^2 + \vec{\pi}^2\right)^2$$

full Lagrangian for $SU(2)_L \otimes SU(2)_R$ σ-model.

(Gell-Mann & Levi, 1960)

As usual write $q^N = q_L^N + q_R^N \Rightarrow$

$$\bar{q}^N i\gamma\cdot\partial q^N = \bar{q}_L^N\, i\gamma\cdot\partial q_L^N + \bar{q}_R^N\ i\gamma\cdot\partial q_R^N$$

$$\bar{q}^N\left[\sigma \mathbb{1} + i\vec{\tau}\cdot\vec{\pi}\,\gamma_5\right] q^N = \Big(\underbrace{\bar{q}^N\frac{\mathbb{1}+\gamma_5}{2}}_{\bar{q}_L} + \underbrace{\bar{q}^N\frac{\mathbb{1}-\gamma_5}{2}}_{\bar{q}_R}\Big)\cdot$$

$$\left[\sigma\mathbb{1} + i\,\vec{\tau}\cdot\vec{\pi}\,\gamma_5\right]\Big(\underbrace{\frac{\mathbb{1}-\gamma_5}{2}q^N}_{q_L} + \underbrace{\frac{\mathbb{1}+\gamma_5}{2}q^N}_{q_R}\Big) = \Big|\text{ as } (\gamma_5)^2=1$$

$$= \sigma\left[\bar{q}^N_L q^N_R + \bar{q}^N_R q^N_L\right] + i\left[-\bar{q}^N_R\,\vec{\tau}\cdot\vec{\pi}\,q^N_L + \bar{q}^N_L\,\vec{\tau}\cdot\vec{\pi}\,q^N_R\right]$$

$$= \bar{q}^N_L\,\Sigma\,q^N_R + \bar{q}^N_R\,\Sigma^+\,q^N_L$$

$$\Rightarrow \mathcal{L} = \bar{q}^N_L\, i\gamma\cdot\partial\, q^N_L + \bar{q}^N_R\, i\gamma\cdot\partial\, q^N_R + \frac{1}{4}\operatorname{tr}\left[\partial_\mu\Sigma\,\partial^\mu\Sigma^+\right]$$

$$+\frac{\mu^2}{4}\operatorname{tr}\left[\Sigma\Sigma^+\right] - \frac{\lambda}{16}\left(\operatorname{tr}\left[\Sigma\Sigma^+\right]\right)^2 -$$

$$-g\left[\bar{q}^N_L\,\Sigma\,q^N_R + \bar{q}^N_R\,\Sigma^+\,q^N_L\right]$$

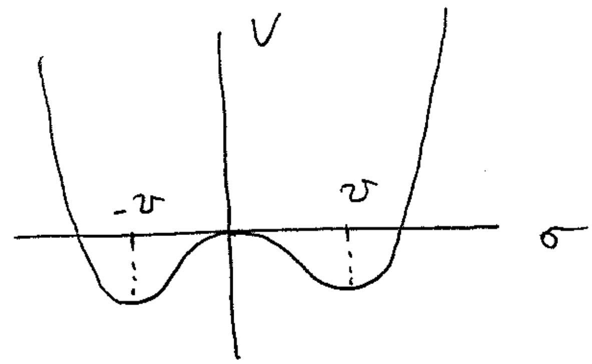
(effective low-energy Lagrangian not QCD, but has the right symmetries)

$\Rightarrow$ this Lagrangian is symmetric under

$$\left[\begin{aligned} q^N_L &\to q^{N\prime}_L = e^{i\vec{\alpha}_L\cdot\frac{\vec{\tau}}{2}}\,q^N_L \equiv U_L\,q^N_L \\ q^N_R &\to q^{N\prime}_R = e^{i\vec{\alpha}_R\cdot\frac{\vec{\tau}}{2}}\,q^N_R \equiv U_R\,q^N_R \\ \Sigma &\to \Sigma' = U_L\,\Sigma\,U_R^+ \end{aligned}\right.$$

$\Rightarrow$ it has $SU(2)_L \otimes SU(2)_R$ symmetry!

For $\mu^2 > 0$ the $SU(2)_L \otimes SU(2)_R$ symmetry is spontaneously broken:



$$\left(\frac{\mu^2}{2}\sigma^2 - \frac{\lambda}{4}\sigma^4\right)' = 0$$

$$\Rightarrow \quad v = \frac{\mu}{\sqrt{\lambda}}$$

$\Rightarrow$ pick $\langle \Psi_0 | \sigma | \Psi_0 \rangle = v$, $\langle \Psi_0 | \vec{\pi} | \Psi_0 \rangle = 0$ as the vacuum.

Write $\sigma = v + \sigma'$ $\Rightarrow$ $\mathcal{L} = \bar{q}^N i \gamma \cdot \partial q^N - g \bar{q}^N [v + \sigma' + i \vec{\tau} \cdot \vec{\pi} \gamma_5] q^N + \frac{1}{2}[\partial_\mu \sigma' \partial^\mu \sigma' + \partial_\mu \vec{\pi} \partial^\mu \vec{\pi}] - \mu^2 \sigma'^2 - \lambda v \sigma' (\sigma'^2 + \vec{\pi}^2) - \frac{\lambda}{4}(\sigma'^2 + \vec{\pi}^2)^2$.

$\Rightarrow$ σ' has mass $\sqrt{2}\mu$.

$\vec{\pi}$ have mass 0. ~ Goldstone bosons (pions)

q^N (proton, neutron) have mass $g v$. ~ can be large!

Identify $\vec{\pi} \Leftrightarrow \bar{q} \gamma_5 \vec{\tau} q$ (q now are real quarks)

$\sigma \Leftrightarrow \bar{q} q$

q^N ~ proton, neutron ~ nucleons

$\Rightarrow$ $SU(2)_L \otimes SU(2)_R$ is spontaneously broken down to $SU(2)$

$\Rightarrow$ pions $^{(\pi^+, \pi^-, \pi^0)}$ are Goldstone bosons of chiral SSB, $m_\pi = 0$ (SU(2) has 3 generators $\Rightarrow$ 3 pions!)

$\Rightarrow$ protons, neutrons get a mass $m_N = g v$ which is large.

$\Rightarrow$ if $SU(2)_L \otimes SU(2)_R$ was exact would have $m_\pi = 0$ but as $m_u \neq m_d \neq 0$ $SU(2)_L \otimes SU(2)_R$ is explicitly broken too $\Rightarrow$ get massive pions!

$\Rightarrow$ for $N_f = 3$ have $SU(3)_L \otimes SU(3)_R$ broken down spontaneously to SU(3) flavor.

$\Rightarrow$ SU(3) has 8 symmetry charges

Q^a, $a = 1, \ldots, 8$

$\Rightarrow$ have 8 Goldstone bosons:

$\pi^+, \pi^-, \pi^0, K^+, K^0, \bar{K}^0, K^-, \eta^0$

$\Rightarrow$ $SU(3)_L \otimes SU(3)_R$ is also badly broken explicitly as $m_s \neq m_u \neq m_d \neq 0$ $\Rightarrow$ K's & η are also massive!

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what is v (VEV) in QCD? Remember $\sigma = \bar{q} q$ $\Rightarrow$

$-v = \langle 0 | \bar{q} q | 0 \rangle \simeq -(230 \text{ MeV})^3$ quark condensate or chiral condensate.

$m_\pi^2 \sim (m_u + m_d) \langle 0 | \bar{q} q | 0 \rangle$.