Last time | The Pardox Model (cont'd)

We discussed the optical theorem in terms of its manifestation in DIS:

\[ W_{\mu \nu} = 2 \text{Im}(\Sigma_{\mu \nu}) \]

\[ T^{\mu \nu} = \begin{array}{c}
  \text{propagator:} \\
  \frac{i}{k^2 - m^2 + i\varepsilon} \rightarrow \frac{k}{k^2 - m^2 + i\varepsilon} \\
  2\pi \delta^4(k^2 - m^2) \\
  2\pi \Theta(h^0) \delta(k^2 - m^2) \\
  \text{positive energy.}
\end{array} \]

Indeed,

\[ 2 \text{Im} \left( i \frac{i}{k^2 - m^2 + i\varepsilon} \right) = 2 \pi \delta(k^2 - m^2) \]

\[ W_{\mu \nu} = \begin{array}{c}
  \text{diagram:} \\
  \{ A_{ij}^{\pm}(p, k) \} \\
  \text{Dirac indices}
\end{array} \]
We get

\[ W_{\mu\nu} = \frac{1}{2m_p} \sum_f e_f^2 \int \frac{d^4k}{(2\pi)^4} A_i^{\mu} (p, k) \left[ \delta_m (k+q) \delta_n \right]_{ij} \cdot S \left( (k+q)^2 \right) \]

Using \( Q^2 \gg q^2, k^\pm \) and \( k^+ \gg k^- \) we get

\[ S \left( (k+q)^2 \right) \approx \frac{X}{Q^2} \delta \left( x - \frac{k^+}{p^+} \right) \]

\( \Rightarrow \) Bjorken \( x \) is the light-cone momentum fraction of the struck quark!
if $x$ is small ($\ll 1$) and $Q$ is large

\[\Rightarrow \quad T_{\text{DIS}} \ll T^\Lambda\]

as $2xp/Q^2 \ll p/\mu^\Lambda \Rightarrow 2xM \ll 2m^\Lambda \ll Q^2$

Interaction is "instantaneous".

Define light cone variables:

for vector $V^\mu$ one has $V^+ = \frac{1}{\sqrt{2}} (V^0 + V^3)$

$V = (V^1, V^2)$

$V^- = \frac{1}{\sqrt{2}} (V^0 - V^3)$

(2d transverse vector)

$V_1 \cdot V_2 = V_1^\mu V_2^\mu = V_1^+ V_2^- + V_1^- V_2^+ - V_1^1 V_2^2$

$W^{\mu\nu} = p

\begin{array}{c}
A_{ab} (p, k) \\
\text{Dirac indices}
\end{array}

\begin{array}{c}
e.g. \\
\begin{align*}
p^+ &= \frac{p}{\sqrt{2}} \\
p^- &= \frac{m^2}{2\sqrt{2}p} \\
p^+ &\gg p^-
\end{align*}
\end{array}

\text{as } W^{\mu\nu} = 2 \text{ Im } (i T^{\mu\nu})

T^{\mu\nu} = \begin{array}{c}
W^{\mu\nu} = \begin{array}{c}
\text{cut (Im part of a propagator)}
\end{array}
\end{array}

\text{as } 2 \text{ Im } \frac{1}{k^2-m^2+i\varepsilon} = 2\pi \delta^{(+)}(k^2-m^2)
We write

\[
W_{\mu
u} = \frac{1}{4\pi m_p} \sum_{f} e_f^2 \int \frac{d^4k}{(2\pi)^4} A^f_{ab}(p, k) \left[ \gamma_\mu \gamma_0 (k+q) \gamma_0 \right]_{ab}
\]

where \( A_{ab} \) is the rest of the diagram (see p. 90).

Start calculating assuming that

\[ Q^2 \gg k^2, \quad B \frac{q^2}{p^2} \gg 1, \quad h^+ \gg h^- \quad (IMF) \]

\[
(k + q)^2 = k^2 + 2h^+ q^- + 2h^- q^+ - h^+ h^- - Q^2
\]

\[ q^2 = 0 \Rightarrow q^+ = q^- \Rightarrow \text{as } h^+ \gg h^- \Rightarrow \text{drop } 2h^- q^+ \]

dropping \( h^2, h \cdot q < Q^2 \) get

\[
(k + q)^2 \approx 2h^+ q^- - Q^2
\]

\[
\delta ((k + q)^2) \approx \delta (2h^+ q^- - Q^2) = \delta \left( \frac{h^+}{p^+} - 2p^+ q^+ - Q^2 \right)
\]

as \( p \cdot q = p^+ q^- \Rightarrow \) and \( \chi_{Bj}^2 = \frac{Q^2}{2p \cdot q} \)

\[
\delta ((k + q)^2) \approx \frac{\chi_{Bj}^2}{Q^2} \delta \left( \chi_{Bj} - \frac{h^+}{p^+} \right)
\]

\[
\Rightarrow \chi_{Bj} = \frac{h^+}{p^+}
\]

Feynman \( x = Bjorken x \)

physical meaning: light cone momentum fraction of the struck quark!
\[ \gamma^0 (h + g) = \gamma^+ (h^- + g^-) + \gamma^- (h^+ + g^+) - \gamma^0 (h^+ + g^-) \]

After \( d^4 k \): \( \gamma^+ \rightarrow p^+ \quad \gamma^- \rightarrow p^- \quad \gamma \rightarrow p = 0 \)

\[ \Rightarrow \text{as } p^+ \gg p^- \text{ keep } \gamma^+ \text{ only} \]

\[ \frac{(k^+)^2}{2(k^+ \gamma^+)} = \frac{k^+}{24^+} = \frac{Q^2}{2x} \]

\[ W_{\mu \nu} = \frac{1}{4 m^2_{p \gamma}} \sum_f e_f^2 \int d^4 k \frac{(2\pi)^4}{A_{a b} (p, k)} \left[ \delta^\mu \delta^\nu \delta^+ \delta_- \right]_{b a} \]

\[ \delta \left( x - \frac{k^+}{p^+} \right) \text{ (see } p^\gamma \text{ decay) } \]

\[ \text{Symmetrize, as } W_{\mu \nu} \text{ is symmetric} \]

Concentrate on \( W_{ij} \sim \frac{1}{2} \left[ \delta^i \delta^j \delta^+ + \delta^i \delta^j \delta^- \right] = \]

\[ = -\frac{1}{2} \delta^+ \{ \delta^i, \delta^j \} = -g_{ij} \delta^+ \quad \text{(we used } W_{ij} = W_{ij}) \]

\[ W_{ij} \sim g_{ij} \text{ from diagram calculations} \]

On the other hand, since \( p = 0 \)

\[ W_{ij} = W_1 (g_{ij} - \frac{g_i g_j}{g^2}) + \frac{w_2}{m^2_p} g_i g_j \left( \frac{p \cdot g}{g^2} \right)^2 = \]

\[ = -W_1 g_{ij} + \frac{g_i g_j}{g^2} \left[ W_1 + \frac{w_2}{m^2_p} (p \cdot g)^2 \right] \sim g_{ij} \]

**Mueller vertex**
\[ W_1 + \frac{W_2}{m_p^2} \frac{(p \cdot q)^2}{q^2} = 0 \]

as \( V = \frac{p \cdot q}{m_p} \) and \( x = \frac{Q^2}{2p \cdot q} = -\frac{q^2}{2p \cdot q} \).

We write

\[ UVW_2 = 2m_p \times W_1 \]

Callan-Gross Relation 169

follows from spin-\( \frac{1}{2} \) nature of quarks!

(would be different for particles with different spin); equivalently:

\[ F_2(x, Q^2) = 2 \times F_1(x, Q^2) \]

Exercise: show that Callan-Gross relation leads to

\[ \frac{d\sigma}{d^3p} \sim \left[ 1 + (1 - \frac{1}{2})^2 \right] W_1 \]

CG relation leads to

\( UVW_2 = 2m_p \times W_1 = \frac{1}{y_f} x \sum_f \frac{2}{(2\pi)^4} \int d^4k A_a^f (p, k) \cdot \left[ (\delta^+)_{ba} \delta(x - \frac{\vec{k} +}{p^+}) \right] \)

\( \Rightarrow \) defining quark distribution:

\[ q_f^c(x) \equiv \frac{1}{2p^+} \int d^4k A_a^f (p, k) (\delta^+)_{ba} \delta(x - \frac{\vec{k} +}{p^+}) \]

we get

\[ F_2 = \sum \frac{\epsilon_f^2}{x(Q^2)^+} q_f^c(x) \]

no \( Q^2 \)-dependence

only \( x \)-dependent

Bjorken scaling (see attached)
Bjorken scaling was first measured at SLAC in 1968; it killed string models and brought back field theories.

\[
F_2 (x) = \sum_f e_f^2 \times q_f (x)
\]

\[
F_1 = \frac{F_2}{2x} = \frac{1}{2} \sum_f e_f^2 q_f (x)
\]

\(F_1\) counts the number of quarks in the proton with the longitudinal momentum fraction = \(x\) (weighted by \(\frac{1}{2} e_f^2\)).

\(F_2\) gives the average \(x\) carried by quarks (weighted by \(e_f^2\)) \& # of quarks at \(x\).
Deep inelastic scattering

Fig. 2.7. Compilation of the world $F_2$ data for DIS on a proton. The proton $F_2$ structure function is plotted as a function of $Q^2$ for a range of values of $x$, as indicated next to the data. It can be seen that, except for very small $x$, $F_2$ is independent of $Q^2$, a manifestation of Bjorken scaling. (We thank Kunihiro Nagano for providing us with this figure.) A color version is available online at www.cambridge.org/9780521112574.

In Fig. 2.7 we show a summary of the world knowledge of the proton $F_2$ structure function. This structure function is plotted as a function of $Q^2$ for many different fixed values of Bjorken-$x$. One can clearly see that, when $x$ is not too small, $F_2$ is independent of $Q^2$. This is the experimental manifestation of Bjorken scaling. We see that the theory we
\[ q^f(x) = \frac{i}{2p^+} \delta^{+5}
\]

\[ \Rightarrow \text{often } p^+(x) \]

is denoted \( q(x) \).

\[ q^f(x, Q^2) \text{ counts } \# \text{ of quarks with light cone momentum } x \text{ and transverse momentum } k_T \leq Q. \]

A parton distribution function \( q^f \sim a^+u \)

\[ \Rightarrow \text{ for a free quark } A^f_{a b}(p, k) \frac{d^4e}{s_{ba}} = \delta^4(p-k) \left( \frac{s}{2\pi} \right)^4 \]

\[ \frac{d^4e}{s_{ba}} = 2p^+ \left( \frac{s}{2\pi} \right)^4 \delta^4(p-k) \Rightarrow (\overline{u}^f_{\text{quark}} = \delta(x-1)) \]

\[ \text{one quark at } x=1 \]

QCD Improved Parton Model: DGLAP equation

How about corrections like \( ?? \)?

These are important corrections.

However, let us first discard the negligible diagrams like