

Last time | finished talking about the Parton Model

=> derived Callan-Gross relation:

$$F_2(x) = 2x F_1(x)$$

note: this is specific to spin- $\frac{1}{2}$ quarks
(it would be different for particles of diff. spin)

Defined quark distribution function:

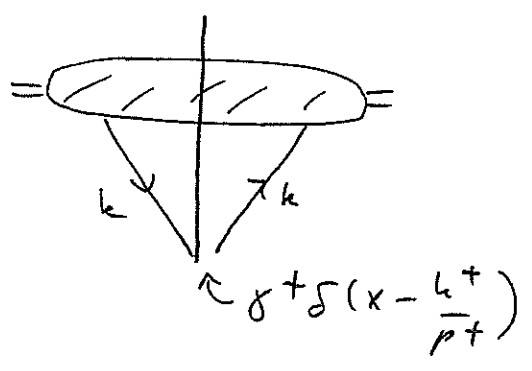
$$g^f(x) \equiv \frac{1}{2p^+} \int \frac{d^4k}{(2\pi)^4} A_{\alpha\beta}^f(p, k) (\delta^+)_{\beta\alpha} S(x - \frac{k^+}{p^+})$$

Obtained

$$F_2(x) = \sum_f e_f^2 x g^f(x)$$
$$F_1(x) = \frac{1}{2} \sum_f e_f^2 g^f(x)$$

F_1, F_2 are independent of Q^2 , and depend only on x :
this is Bjorken scaling.

$$g^f(x) = \frac{1}{2p^+}$$



\Rightarrow often $p^f(x)$ is denoted $g(x)$.

$g^f(x, Q^2)$ counts # of quarks with light cone momentum x and transverse momentum $k_T \leq Q$.

parton distribution function ($g^f \sim a^+ a^-$)

\Rightarrow for a free quark $A_{ab}^f(p, k) \delta_{ba}^+ = \delta^4(p-k) \cdot (2\pi)^4$

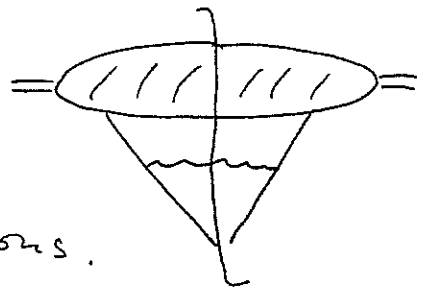
$$\frac{\bar{u}_b(p) \delta_{ba}^+ u_a(p)}{= 2p^+} = 2p^+ (2\pi)^4 \delta^4(p-k) \Rightarrow \boxed{g_{quark}^f(x) = \delta(x-1)}$$

one quark at $x=1$

Reskin, ch. 17.5

terman d.14 QCD Improved Parton Model: DGLAP equation

IK, Levin ch. 2.4

How about corrections like  ?

These are important corrections.

However, let us first discard the negligible



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Work in Light Cone (LC)

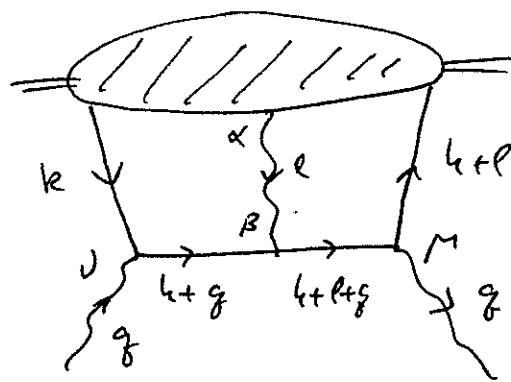
(94)

gauge: $\eta \cdot A = A^+ = 0$

$(\eta^+ = 0, \eta^- = 1, \eta^\perp = 0)$

$Q^2 \sim$ very large:

$$\Gamma_{\mu\nu\beta} = \delta_\mu \frac{\delta_\nu(k+l+q)}{(k+l+q)^2 + i\epsilon} \delta_\beta$$



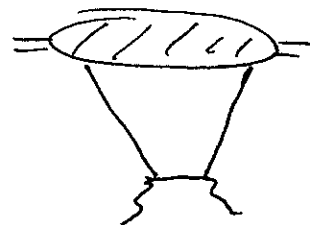
$$\frac{\delta_\nu(k+q)}{(k+q)^2 + i\epsilon} \delta_\nu ; \text{ Now, } (q+l)^2 = q^2 + 2k \cdot q = -Q^2 + 2k^+ q^-$$

as k^+ is large. Similarly $(k+l+q)^2 \simeq -Q^2 +$

$$+ 2q^-(k^+ + l^+)$$

$$\Rightarrow \Gamma_{\mu\nu\beta} = \frac{1}{Q^4} \frac{\delta_\mu \delta_\nu(k+l+q) \delta_\beta \delta_\nu(k+q) \delta_\nu}{\left(1 - \frac{2k^+ q^-}{Q^2} - i\epsilon\right) \left(1 - \frac{2(k^+ + l^+) q^-}{Q^2} - i\epsilon\right)}$$

Seems like $\Gamma_{\mu\nu\beta} = O\left(\frac{1}{Q^4}\right)$, which is suppressed compared to $O\left(\frac{1}{Q^2}\right)$ diagram



However, when integrating over dl^+ may

(pick up the pole at $l^+ = -k^+ + \frac{Q^2}{2q^-}$,

getting a Q^2 in the numerator.

$$\gamma \cdot (k+q) \approx \gamma^+ (k^- + q^-) + \gamma^- (k^+ + q^+) - \underline{\gamma} \cdot (k+q) \quad (72)$$

$$\gamma \cdot (k+l+q) = \gamma^+ (k^- + l^- + q^-) + \gamma^- (k^+ + l^+ + q^+) - \underline{\gamma} \cdot (k+l+q) \quad ($$

(or integrating over k^+ picking up a pole)

\Rightarrow When taking Im part, get $2k^+ q^- = Q^2 \Rightarrow$

$$\Rightarrow q^- = \frac{Q^2}{2k^+} \Rightarrow \text{2nd denominator becomes}$$

$$Q^2\text{-independent: } 1 - \frac{2(k^+ + l^+) q^-}{Q^2} = 1 - \frac{k^+ + l^+}{k^+} = -\frac{l^+}{k^+}.$$

\Rightarrow keep only q^- terms in the numerator (Q^2 -dep)

$$\Rightarrow \gamma \cdot (k+q) \approx \gamma^+ q^-, \quad \gamma \cdot (k+l+q) \approx \gamma^+ q^-$$

$$\Rightarrow \Gamma_{\mu\nu\beta} \sim \gamma_\mu^+ \gamma_\nu^+ \gamma_\beta^+ + \gamma_\mu^+ \gamma_\nu^+ \gamma_\beta^+$$

$$\Rightarrow \gamma^+ \gamma_\beta^+ \gamma^+ \text{ is } \neq \text{ only if } \beta = "-" \text{ as } (\gamma^+)^2 =$$

$$= \left(\frac{\gamma^0 + \gamma^3}{\sqrt{2}} \right)^2 = \frac{1}{2} \left((\gamma^0)^2 + (\gamma^3)^2 + \{\gamma^0, \gamma^3\} = 0 \right) = \frac{1}{2} (1-1) = 0.$$

But if $\beta = - \Rightarrow$ need $D^{\alpha+}(e)$

$$D_{\alpha\beta}(e) = \frac{-i}{e^2} \left[g_{\alpha\beta} - \frac{\gamma_\alpha^+ \ell_\beta^+ + \gamma_\beta^+ \ell_\alpha^+}{\gamma \cdot e} \right]$$

$$D^{\alpha+} = \frac{-i}{e^2} \left[g^{\alpha+} - \frac{\gamma^\alpha \ell^+}{e^+} \right] = 0 \Rightarrow \text{never get } Q^2 \text{ ($$

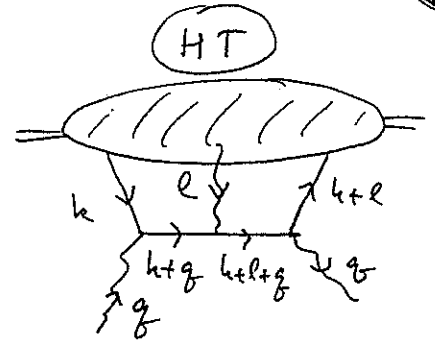
in the numerator $\Rightarrow O\left(\frac{1}{Q^2}\right)$ "Higher Twist"

Let us work in Light Cone

gauge defined by

$$\eta \cdot A = A^+ = 0$$

$$(\eta^+ = 0, \eta^- = 1, \eta^i = 0)$$



In DIS Q^2 is very large such that

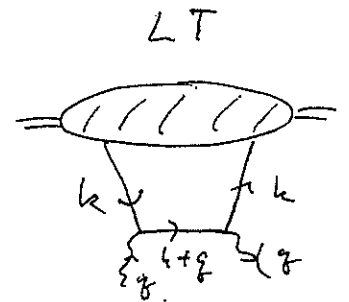
$$\frac{|k^2|}{Q^2} \ll 1, \quad \frac{|l^2|}{Q^2} \ll 1$$

$$\Rightarrow \text{approximate } (k+q)^2 \approx q^2 \approx -Q^2$$

$$(k+l+q)^2 \approx q^2 \approx -Q^2$$

$$\Rightarrow \text{diagram HT} \sim \frac{1}{Q^4}$$

Compare with leading parton model diagram $LT \sim \frac{1}{Q^2}$



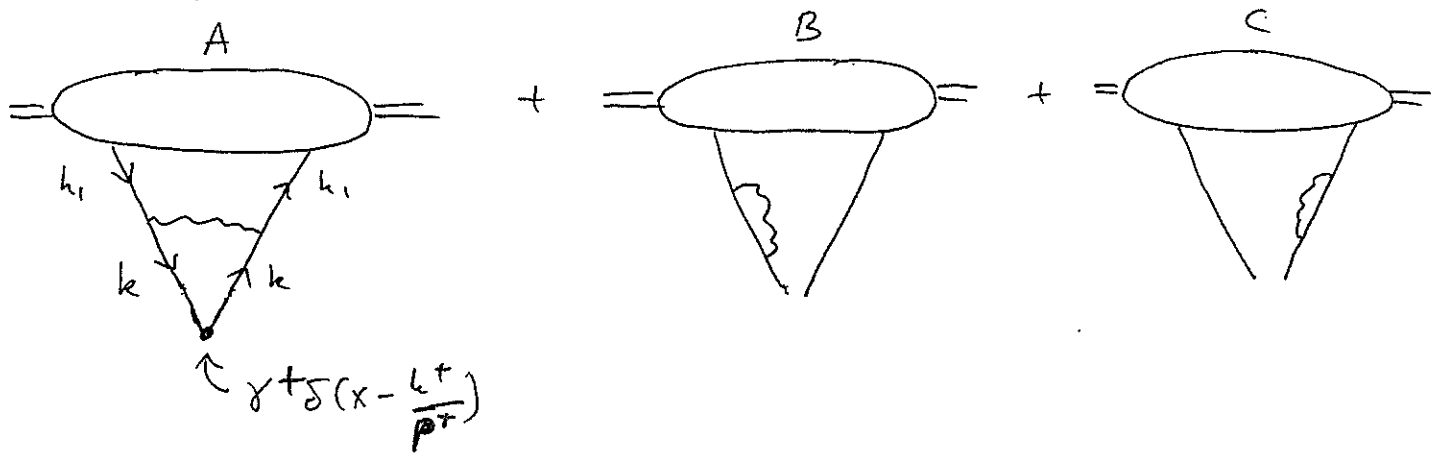
$$\Rightarrow HT \ll LT \text{ as for large } Q^2: \frac{1}{Q^4} \ll \frac{1}{Q^2}$$

HT stand for Higher Twist $\sim 1/Q^4$

LT - Leading Twist $\sim 1/Q^2$

Multiple rescatterings are Higher Twists, usually suppressed by $1/Q^2$ (A - some small scale)
(true in LC gauge only!)

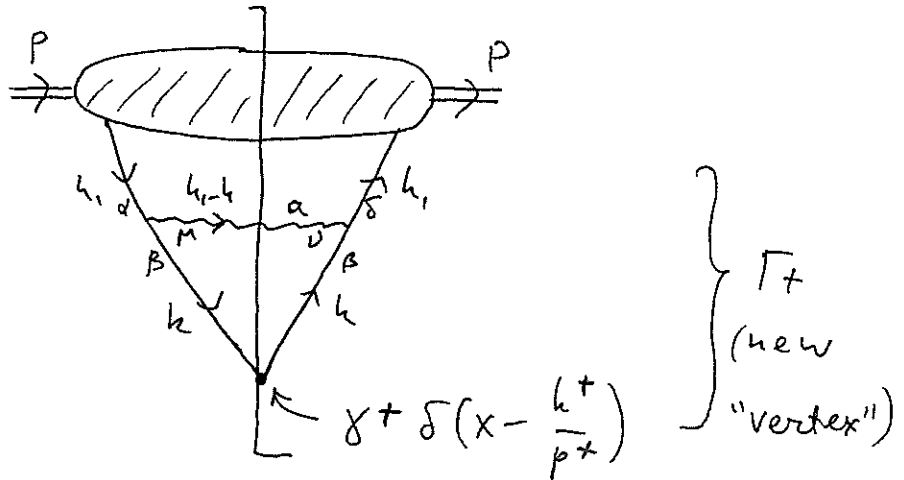
We need to calculate the following corrections ⁽¹⁾ to the parton model:



We will work out diagram A only: $|\underline{k}\rangle \gg |\underline{k}_1|$

$$g^f(x, Q^2) = \frac{1}{2p^+}$$

$$(T^a T^a)_{\delta\alpha} = \delta_{\alpha\delta} \frac{N_c^2 - 1}{2N_c}$$



$$\Gamma^+ = (ig)^2 \overbrace{(T^a)_{\delta\beta} (T^a)_{\beta\alpha}} \int \frac{d^4 h}{(2\pi)^4} \gamma^\nu \frac{i\gamma_0 k}{k^2} \gamma^+ \frac{i\gamma \cdot k}{k^2} \gamma^\mu$$

$$\delta(x - \frac{h^+}{p^+}) (-2\pi) \delta((h_+ - h)^2) \left[g_{\mu\nu} - \frac{\gamma_\mu (h_\nu - k_\nu) + \gamma_\nu (h_\mu - k_\mu)}{h_+^+ - h^+} \right]$$

where we used the fact that gluon propagator in the $\gamma \cdot A = A^+ = 0$ light cone gauge is

$$D_{\mu\nu}(l) = \frac{-i}{l^2} \left[g_{\mu\nu} - \frac{\gamma_\mu l_\nu + \gamma_\nu l_\mu}{\gamma \cdot l} \right] \text{ with } \frac{-i}{l^2} \rightarrow -2\pi \delta(l^2)$$

and $\gamma \cdot l = l^+$.

First integrate over k_- :

$$\int dk^- \delta((k_1 - k)^2) = \frac{1}{2(k_1 - k)^+} \text{ with } k^- = k_1^- - \frac{(k_1 - k)^2}{2(k_1 - k)^+}$$

Also, k^+ -integration is easy: $\int dk^+ \delta(x - \frac{k^+}{p^+}) = p^+$

Defining $d_s \equiv \frac{g^2}{4\pi}$ we write ($c_F \equiv \frac{N_c^2 - 1}{2N_c}$)

$$\Pi^+ = - \frac{d_s c_F}{4\pi^2} \delta_{\alpha\beta} \int \frac{d^2 k}{k^4} \gamma^\nu \gamma_{\cdot k} \gamma^+ \gamma_{\cdot k} \gamma^\mu \left[g_{\mu\nu} - \frac{\gamma_\mu (k_{1\nu} - k_\nu) + \gamma_\nu (k_{1\mu} - k_\mu)}{k_1^+ - k^+} \right] \frac{p^+}{(k_1 - k)^+}$$

Evaluate k^2 : $k^2 = 2k^+k^- - \underline{k}^2 = 2k^+ \left(k_1^- - \frac{(k_1 - k)^2}{2(k_1 - k)^+} \right) - \underline{k}^2 =$

$$= \left| \text{define } z = \frac{k^+}{k_1^+} = 2z k_1^+ k_1^- - \frac{z}{1-z} (k_1 - k)^2 - \underline{k}^2 = \right.$$

$$= z k_1^2 + z \underline{k}_1^2 - \frac{z}{1-z} (\underline{k}_1^2 - 2\underline{k}_1 \cdot \underline{k} + \underline{k}^2) - \underline{k}^2 =$$

$$= z k_1^2 - \frac{1}{1-z} (\underline{k} - z \underline{k}_1)^2 \approx - \frac{\underline{k}^2}{1-z} \text{ for } k_\perp \gg k_{1\perp}, \underline{k}^2 \gg k_1^2$$

Evaluate $\gamma^\nu \gamma_{\cdot k} \gamma^+ \gamma_{\cdot k} \gamma^\mu \left[g_{\mu\nu} - \frac{\gamma_\mu (k_1 - k)_\nu + \gamma_\nu (k_1 - k)_\mu}{k_1^+ - k^+} \right]$

First note that as $\{\gamma_\rho, \gamma_\sigma\} = 2g_{\rho\sigma}$

$$\gamma_{\cdot k} \gamma^+ \gamma_{\cdot k} = \gamma_{\cdot k} \left[\underbrace{\{\gamma^+, \gamma_{\cdot k}\}}_{2k^+} - \gamma_{\cdot k} \gamma^+ \right] = 2k^+ \gamma_{\cdot k} - k^2 \gamma^+ \quad \textcircled{2} \quad \textcircled{1}$$

Let's put ① and ② back into the monster ⁽⁷⁷⁾

expression:

$$\textcircled{1} = -k^2 \delta^\nu \delta^+ \delta^M \left[g_{\mu\nu} - \frac{\cancel{\gamma}^\mu (h, -h)_\nu + \cancel{\gamma}^\nu (h, -h)_\mu}{\cancel{\gamma} \cdot (h, -h)} \right] =$$

$$= (\text{as } \delta^{+2} = 0) = -k^2 \delta_\mu \delta^+ \delta^M = 2 \delta^+ k^2$$

since $\delta_\mu \delta_\alpha \delta^M = -2 \delta_\alpha$

$$\textcircled{2} = 2 h^+ \delta^\nu \delta \cdot k \delta^M \left[g_{\mu\nu} - \frac{\cancel{\gamma}^\mu (h, -h)_\nu + \cancel{\gamma}^\nu (h, -h)_\mu}{\cancel{\gamma} \cdot (h, -h)} \right] =$$

$$= 2 h^+ \left[\delta_\mu \delta \cdot k \delta^M - \frac{1}{\cancel{\gamma} \cdot (h, -h)} \left(\delta \cdot (h, -h) \delta \cdot k \delta^+ + \delta^+ \delta \cdot k \delta \cdot (h, -h) \right) \right] = 2 h^+ \left[-2 \delta \cdot k - \frac{1}{\cancel{\gamma} \cdot (h, -h)} \cdot \right.$$

$$\left. \left(-2 k^2 \delta^+ + \delta \cdot h_1 \delta \cdot k \delta^+ + \delta^+ \delta \cdot k \delta \cdot h_1 \right) \right]$$

we want to swap and move over here.

$$\textcircled{2} = 2 k^+ \left[-2 \delta \cdot k - \frac{1}{\cancel{\gamma} \cdot (h, -h)} \left(-2 k^2 \delta^+ + 2 h \cdot h_1 \delta^+ + 2 k^+ \delta \cdot h_1 - 2 h_1^+ \delta \cdot k \right) \right]$$

Now we are interested in the regime where

$$|k| \gg |h_1|, \quad \underline{k}^2 \gg h_1^2, \quad \text{i.e. } |k| \text{ is VERY LARGE.}$$

at the leading order in $|k|$:

$$h^- \approx - \frac{k^2}{2 k^+ (1-z)}$$

For large $|k|$: ① $\approx -2\gamma^+ \underline{k}^2 \frac{1}{1-z}$

as $k^2 = z k_1^2 - \frac{1}{1-z} (\underline{k} - z \underline{k}_1)^2 \rightarrow -\frac{\underline{k}^2}{1-z}$

$$\textcircled{2} \approx 2z \left[+ \cancel{\gamma^+} \frac{\underline{k}^2}{\cancel{2(1-z)}} - \frac{1}{1-z} \left(\frac{2\underline{k}^2}{1-z} \gamma^+ - \frac{\underline{k}^2}{1-z} \gamma^+ + \right. \right.$$

$\left. - 2k_1^+ \gamma^+ \right]$ ss

$$+ \underbrace{\frac{\underline{k}^2}{1-z} \gamma^+}_{\approx -2k_1^+ \gamma^+} \left. \right] = 2z \gamma^+ \frac{\underline{k}^2}{1-z} \left[1 - \frac{2}{1-z} \right] =$$

$$= -2\gamma^+ \underline{k}^2 \frac{z(1+z)}{(1-z)^2}$$

$$\Rightarrow \textcircled{1} + \textcircled{2} = -2\gamma^+ \underline{k}^2 \frac{1+z^2}{(1-z)^2}$$

Plugging it all back we get

$$\Gamma^+ = -\frac{\alpha_s C_F}{4\pi^2} \int_{\alpha\delta} \int \frac{d^2k}{k^4} \underbrace{(1-z)^2}_{(4k^2)^2} (-2)\gamma^+ \underline{k}^2 \frac{1+z^2}{(1-z)^2} \frac{p^+/k_1^+}{1-z}$$

\Rightarrow defining Bjorken (or Feynman) x for quark k_1

as $x_1 \equiv \frac{k_1^+}{p^+}$ we get

$$\Gamma^+ = \gamma^+ \frac{1}{x_1} \frac{\alpha_s C_F}{2\pi} \int \frac{d\underline{k}^2}{\underline{k}^2} \frac{1+z^2}{1-z}$$

$$\Gamma^+ = \gamma^+ \frac{1}{x_1} \frac{\alpha_s C_F}{2\pi} \int_{\underline{k}_1^2}^{Q^2} \frac{d\underline{k}^2}{\underline{k}^2} \frac{1+z^2}{1-z} \sim \text{putting the proper integration limits in}$$

$$\Gamma^+ \sim \alpha_s \cdot \ln(Q^2/\underline{k}_1^2) \sim \alpha_s \ln Q^2/\Lambda^2$$

$\alpha_s \ll 1$ (perturbation theory, small coupling)

$\ln(Q^2/\Lambda^2) \gg 1$ (DIS with large Q^2)

$\alpha \ln \frac{Q^2}{\Lambda^2} \sim 1$ our resummation parameter!

Leading Logarithmic Approximation

Remember: we neglected terms suppressed

by $\frac{\underline{k}_1^2}{\underline{k}^2}, \frac{\underline{k}_1^4}{\underline{k}^4}, \dots \Rightarrow$ they give

$$\int_{\underline{k}_1^2}^{Q^2} \frac{d\underline{k}^2}{\underline{k}^4} \underline{k}_1^2 \sim \left(\frac{1}{\underline{k}_1^2} - \frac{1}{Q^2} \right) \underline{k}_1^2 \sim 1 - \frac{\underline{k}_1^2}{Q^2}$$

\uparrow no log \uparrow higher twist

Old (LO) Parton Model vertex (Mueller vertex)

was $\gamma^+ \delta(x - \frac{k^+}{p^+}) \sim$ same γ^+ matrix as Γ^+

$$\Rightarrow Q^2 \frac{\partial}{\partial Q^2} g^f(x, Q^2) = \frac{\alpha}{2\pi} \int_x^1 \frac{dx_1}{x_1} P_{ff} \left(\frac{x}{x_1} \right) g^f(x_1, Q^2)$$