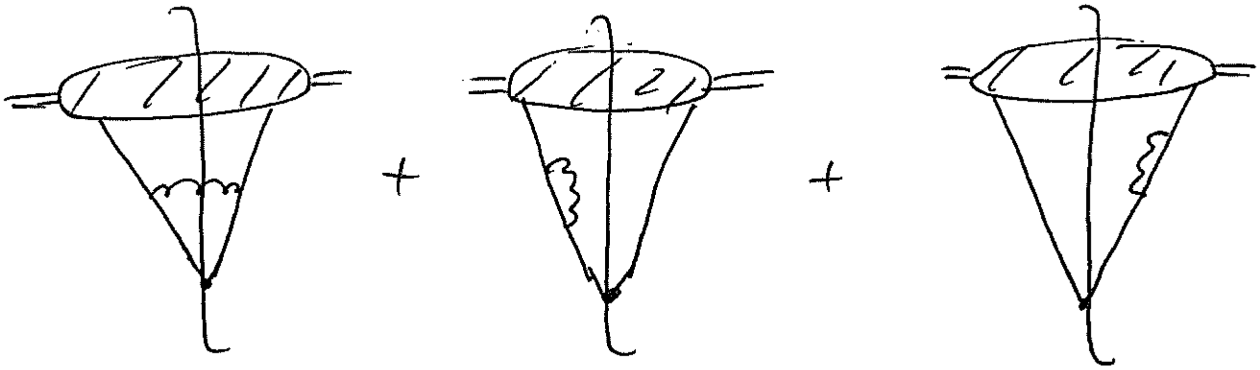
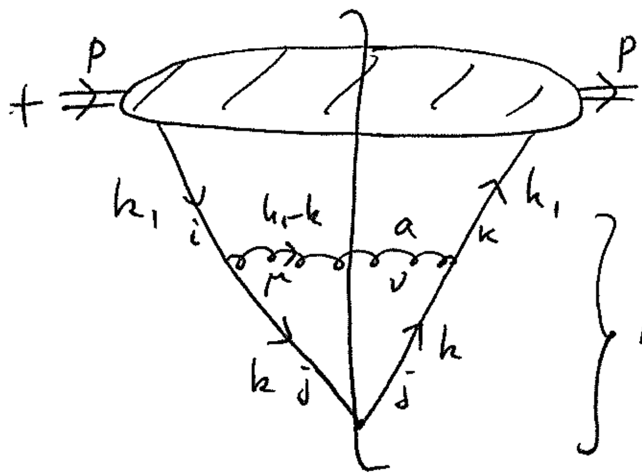
Last time | QCD-Improved Parton Model: DGLAP Equation (cont'd)



QCD corrections to $q^f(x)$.

Concentrated on the real correction:



$A^+ = 0$ light-cone gauge

$A^+ = \eta \cdot A = 0$, $\eta^\mu = (\overset{+}{0}, \overset{-}{1}, \underline{\overset{\perp}{0}})$

$\Gamma^+ \sim$ effective (renormalized) vertex

$$\Gamma^+ = -\frac{\alpha_s C_F}{4\pi^2}\,\delta_{ik}\int \frac{d^2k_\perp}{k^4}\,\underbrace{\gamma^\nu \not{k}\gamma^+\not{k}\gamma^\mu\left[g_{\mu\nu} - \frac{\eta_\mu(k_1-k)_\nu + \eta_\nu(k_1-k)_\mu}{k_1^+ - k^+}\right]}_{=\,-2\gamma^+ k_\perp^2 \frac{1+z^2}{(1-z)^2}}$$

$\cdot\ \dfrac{p^+}{k_1^+ - k^+}$

after some algebra we arrived at this

$k^2 \simeq \dfrac{-k_\perp^2}{1-z}$ where $z = k^+/k_1^+$

We assumed transverse momentum ordering:

$$\boxed{Q^2 \gg k_\perp^2 \gg \underline{k}_1^2,\ k_1^2 \gg \Lambda_{QCD}^2}$$

For large $|\underline{k}|$: ① $\approx -2\gamma^+ \underline{k}^2 \frac{1}{1-z}$

as $\underline{k}^2 = z\underline{k}_1^2 - \frac{1}{1-z}(\underline{k} - z\underline{k}_1)^2 \to -\frac{\underline{k}^2}{1-z}$

$-2k^* k_1^+ \gamma^+$

ss

② $\approx 2z\Big[+\cancel{z}\gamma^+\frac{\underline{k}^2}{\cancel{z}(1-z)} - \frac{1}{1-z}\Big(\frac{2\underline{k}^2}{1-z}\gamma^+ - \cancel{\frac{\underline{k}^2}{1-z}\gamma^+} +$

$+ \underbrace{\cancel{\frac{\underline{k}^2}{1-z}\gamma^+}}_{\approx -2k_1^+\gamma^+ k^-}\Big)\Big] = 2z\,\gamma^+\frac{\underline{k}^2}{1-z}\Big[1 - \frac{2}{1-z}\Big] =$

$= -2\gamma^+\underline{k}^2 \frac{z(1+z)}{(1-z)^2}$

We assume transverse momentum ordering:

$\boxed{Q^2 \gg k_\perp^2 \gg \underline{k}_1^2, k_1^2 \gg \Lambda^2_{QCD}}$

$\Rightarrow$ ① + ② $= -2\gamma^+\underline{k}^2 \frac{1+z^2}{(1-z)^2}$

Plugging it all back we get

$$\Gamma^+ = -\frac{\alpha_s C_F}{4\pi^2}\delta_{\alpha\delta}\int \frac{d^2k}{\underbrace{\underline{k}^4}_{(\underline{k}^2)^2}}(1-z)^2(-2)\gamma^+\underline{k}^2\frac{1+z^2}{(1-z)^2}\cdot\frac{p^+/k_1^+}{1-z}$$

$\Rightarrow$ defining Bjorken (or Feynman) x for quark k_1

as $x_1 \equiv \frac{k_1^+}{p^+}$ we get

$$\boxed{\Gamma^+ = \gamma^+\frac{1}{x_1}\delta_{\alpha\delta}\frac{\alpha_s C_F}{2\pi}\int\frac{d\underline{k}^2}{\underline{k}^2}\frac{1+z^2}{1-z}}$$

$$\Gamma^{+} = \gamma^{+} \frac{1}{x_1} \delta_{\alpha s} \frac{\alpha_s C_F}{2\pi} \int_{\underline{k}_1^2}^{Q^2} \frac{d\underline{k}^2}{\underline{k}^2} \frac{1+z^2}{1-z}$$ ~ putting the proper integration limits in

$$\Gamma^{+} \sim \alpha_s \cdot \ln\left(Q^2/\underline{k}_1^2\right) \sim \alpha_s \ln Q^2/\Lambda^2$$

$\alpha_s << 1$ (perturbation theory, small coupling)

$\ln\left(Q^2/\Lambda^2\right) >> 1$ (DIS with large Q^2)

$\alpha \ln \frac{Q^2}{\Lambda^2} \sim 1$ our resummation parameter!

"<u>Leading</u> <u>Logarithmic</u> <u>Approximation</u>"

Remember: we neglected terms suppressed by $\frac{\underline{k}_1^2}{\underline{k}^2}, \frac{\underline{k}_1^4}{\underline{k}^4}, \ldots \Rightarrow$ they give

$$\int_{\underline{k}_1^2}^{Q^2} \frac{dk^2}{k^4} \underline{k}_1^2 \sim \left(\frac{1}{\underline{k}_1^2} - \frac{1}{Q^2}\right) \underline{k}_1^2 \sim 1 - \frac{\underline{k}_1^2}{Q^2}$$

no log ↑ ↖ higher twist

Old (LO) Parton Model vertex (Mueller vertex)

was $\gamma^{+} \delta\left(x - \frac{k^+}{p^+}\right)$ ~ same γ_+ matrix as Γ_+

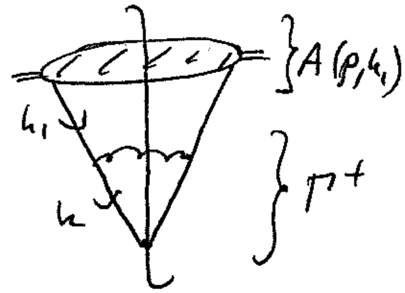
$$\Rightarrow \boxed{Q^2 \frac{\partial}{\partial Q^2} q^f(x, Q^2) = \frac{\alpha}{2\pi} \int_x^1 \frac{dx_1}{x_1} P_{qq}\left(\frac{x}{x_1}\right) q^f(x_1, Q^2)}$$

Start with the definition of quark PDF,

$$q^f(x,Q^2) = \frac{1}{2p^+}\int \frac{d^4k}{(2\pi)^4} A^f_{\alpha\beta}(p,k)\left(\gamma^+\right)_{\beta\alpha}\,\delta\!\left(x-\frac{k^+}{p^+}\right)$$

$\Downarrow$ correction is obtained by replacing $\gamma^+\delta(x-\frac{k^+}{p^+}) \to \Gamma^+$

$$\delta q^f(x,Q^2) = \frac{1}{2p^+}\int \frac{d^4k_1}{(2\pi)^4} A^f_{\alpha\beta}(p,k_1)\left(\Gamma^+\right)_{\beta\alpha}$$



$\Downarrow$ plug in Γ^+ we found:

$$\delta q^f(x,Q^2) = \frac{1}{2p^+}\int \frac{d^4k_1}{(2\pi)^4} A^f_{\alpha\beta}(p,k_1)\left(\gamma^+\right)_{\beta\alpha}\frac{p^+}{k_1^+}\cdot\frac{\alpha C_F}{2\pi}\int_{\underline{k}_1^2}^{Q^2}\frac{dk_\perp^2}{k_\perp^2}\left(\frac{1+z^2}{1-z}\right)_+$$

Rewrite $\frac{p^+}{k_1^+} = \int_x^1 \frac{dx_1}{x_1}\,\delta\!\left(x_1-\frac{k_1^+}{p^+}\right)$ with x_1 a dummy variable.

(Note that $k_1^+ > k^+ \Rightarrow \frac{k_1^+}{p^+} > x \Rightarrow 1 > x > x_1$ is the right range of integration.)

$$\delta q^f(x,Q^2) = \int_x^1 \frac{dx_1}{x_1}\cdot\overbrace{\frac{1}{2p^+}\int\frac{d^4k_1}{(2\pi)^4}A^f_{\alpha\beta}(p,k_1)\left(\gamma^+\right)_{\beta\alpha}\,\delta\!\left(x_1-\frac{k_1^+}{p^+}\right)}^{q^f(x_1,\,k_\perp^2)\ \text{as}\ \underline{k}_1^2 < k_\perp^2}\cdot\frac{\alpha C_F}{2\pi}\int_{\underline{k}_1^2}^{Q^2}\frac{dk_\perp^2}{k_\perp^2}\left(\frac{1+z^2}{1-z}\right)_+$$

$$\Rightarrow \delta q^f(x,Q^2) = q^f(x,Q^2) - q^f(x,\underline{k}_1^2) = \frac{\alpha C_F}{2\pi}\int_x^1\frac{dx_1}{x_1}\left(\frac{1+z^2}{1-z}\right)_+\cdot\int_{\underline{k}_1^2}^{Q^2}\frac{dk_\perp^2}{k_\perp^2}\,q^f(x_1,k_\perp^2)$$

with $z = \frac{x}{x_1} = \frac{k^+}{k_1^+}$

Differentiating both sides w.r.t. $\frac{\partial}{\partial \ln Q^2}$ we get

$$\boxed{\frac{\partial}{\partial \ln Q^2} q^f(x, Q^2) = \frac{\alpha C_F}{2\pi} \int_x^1 \frac{dx_1}{x_1} \left(\frac{1 + \left(\frac{x}{x_1}\right)^2}{1 - \frac{x}{x_1}} \right)_+ q^f(x_1, Q^2).}$$

Defining $P_{qq}(z) = C_F \left(\frac{1+z^2}{1-z} \right)_+$ we get

$$\boxed{\frac{\partial}{\partial \ln Q^2} q^f(x, Q^2) = \frac{\alpha_s}{2\pi} \int_x^1 \frac{dx_1}{x_1} P_{qq}\left(\frac{x_1}{x}\right) q^f(x_1, Q^2)}.$$

where $x = \frac{k^+}{p^+}$, $x_1 = \frac{k_1^+}{p^+} \Rightarrow z = \frac{k^+}{k_1^+} = \frac{x}{x_1}$.

as $z < 1 \Rightarrow x_1 > x$ in the integral.

Including the virtual terms (B and C)

gives

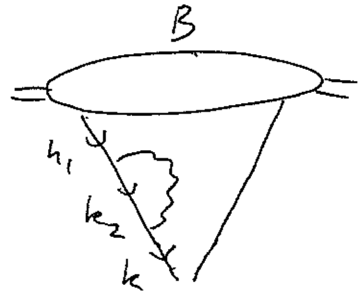
$$P_{qq}(z) = C_F \left(\frac{1+z^2}{1-z}\right)_+$$

~ splitting function

where

$$\int_0^1 dz \, [h(z)]_+ \, f(z) = \int_0^1 dz \, h(z) \, [f(z) - f(1)]$$

Easy to understand:



$\propto \delta(k - k_1)$

$\Downarrow$

$x = x_1$

$\Rightarrow q_f(x_1, Q^2) = q_f(x, Q^2)$

as $x = x_1$

"real" part, diagram A

$$Q^2 \frac{\partial}{\partial Q^2} q^f(x, Q^2) = \frac{\alpha C_F}{2\pi} \int_x^1 \frac{dx_1}{x_1} \frac{1 + (x/x_1)^2}{1 - x/x_1} \cdot \left[q^f(x_1, Q^2) - q^f(x, Q^2) \right]$$

Virtual corrections, graphs B & C

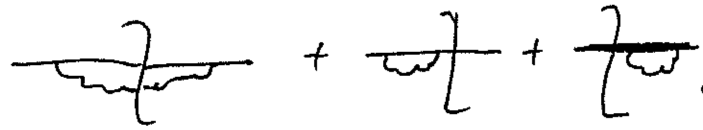
bare quark state $|\psi_0\rangle =$ $\Rightarrow \langle\psi_0|\psi_0\rangle = 1$

(normalization)

dressed quark state $|\psi\rangle =$ 

$|\psi_0\rangle$ $\quad |\psi_1\rangle \quad$ $V|\psi_0\rangle$

normalization:

$\langle\psi|\psi\rangle = 1 = \langle\psi_0|\psi_0\rangle +$ 

$$= 1 + \langle\psi_1|\psi_1\rangle + 2V\langle\psi_0|\psi_0\rangle = 1 + \langle\psi_1|\psi_1\rangle + 2V$$

$$\Rightarrow V = -\frac{1}{2}\langle\psi_1|\psi_1\rangle$$

$\Rightarrow$ graphs $B, C = -\frac{1}{2} A \Rightarrow B + C = -A$.

~ simply imposed probability conservation!

Def. Defining flavor singlet distribution

$$\Sigma(x, Q^2) \equiv \sum_f \left[q^f(x, Q^2) + q^{\bar{f}}(x, Q^2) \right]$$

Def. and flaver non-singlet

$$\Delta^{f\bar{f}}(x, Q^2) \equiv q^f(x, Q^2) - q^{\bar{f}}(x, Q^2)$$

we write

$$\boxed{Q^2 \frac{\partial}{\partial Q^2} \Delta^{f\bar{f}}(x, Q^2) = \frac{\alpha(Q^2)}{2\pi} \int_x^1 \frac{dx_1}{x_1} P_{qq}\left(\frac{x}{x_1}\right) \cdot \Delta^{f\bar{f}}(x_1, Q^2)}$$

and

$$\boxed{Q^2 \frac{\partial}{\partial Q^2} \begin{pmatrix} \Sigma(x, Q^2) \\ G(x, Q^2) \end{pmatrix} = \frac{\alpha(Q^2)}{2\pi} \int_x^1 \frac{dx_1}{x_1} \begin{pmatrix} P_{qq}\left(\frac{x}{x_1}\right) & P_{qG}\left(\frac{x}{x_1}\right) \\ P_{Gq}\left(\frac{x}{x_1}\right) & P_{GG}\left(\frac{x}{x_1}\right) \end{pmatrix} \cdot \begin{pmatrix} \Sigma(x_1, Q^2) \\ G(x_1, Q^2) \end{pmatrix}}$$

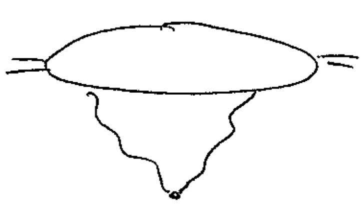
Dokshitzer, Gribov, Lipatov, Altarelli, Parisi

(DGLAP) Equations

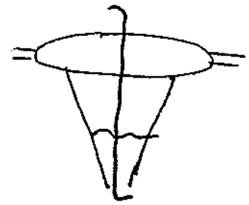
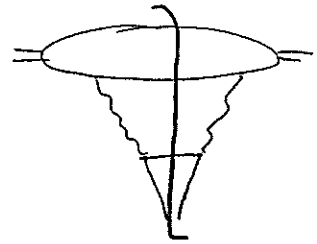
GL ~ QED case ~ '72

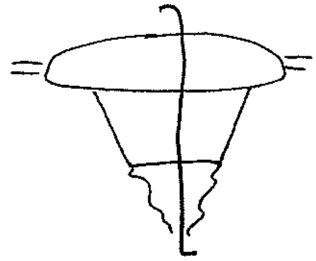
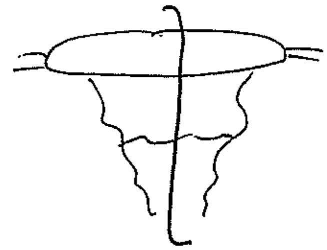
D, A & P ~ QCD case, '77

Def.

$G(x, Q^2) =$  $\sim \langle A_i A_i \rangle$ in $A_+ = 0$ gauge

gluon distribution function

$P_{qq} \sim$  , $P_{qG} =$ 

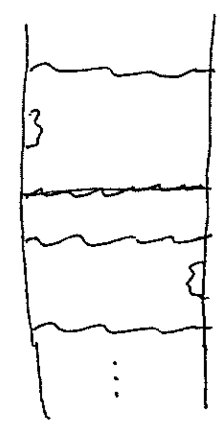
$P_{Gq} \sim$  , $P_{GG} \sim$ 

After explicit calculations one gets the splitting functions:

$$\begin{cases} P_{qq}(z) = C_F \left(\dfrac{1+z^2}{1-z}\right)_+ \\[2ex] P_{Gq}(z) = C_F \, \dfrac{1+(1-z)^2}{z} \\[2ex] P_{qG}(z) = N_f \left[z^2 + (1-z)^2 \right] \\[2ex] P_{GG}(z) = 2N_c \left[\dfrac{z}{(1-z)_+} + \dfrac{1-z}{z} + z(1-z) \right] + \dfrac{11 N_c - 2N_f}{6} \, \delta(z-1) \end{cases}$$

Note that $P_{Gq}(z)$ can be obtained from $P_{qq}(z)$ by substituting $z \to 1-z$ and dropping virtual corrections.

Iterate the evolution for $q_f(x, Q^2)$:



(Defining