Last time | QCD-Improved Parton Model:

DGLAP Equation (cont'd)

after completing a calculation we have arrived at:

$$\frac{\partial}{\partial \ln Q^2} \Delta^{f\bar{f}}(x,Q^2) = \frac{\alpha_s}{2\pi} \int_x^1 \frac{dx_1}{x_1} P_{qq}\left(\frac{x}{x_1}\right) \Delta^{f\bar{f}}(x_1,Q^2)$$

$$\frac{\partial}{\partial \ln Q^2} \begin{pmatrix} \Sigma(x,Q^2) \\ G(x,Q^2) \end{pmatrix} = \frac{\alpha_s}{2\pi} \int_x^1 \frac{dx_1}{x_1} \begin{pmatrix} P_{qq}\left(\frac{x}{x_1}\right) & P_{qG}\left(\frac{x}{x_1}\right) \\ P_{Gq}\left(\frac{x}{x_1}\right) & P_{GG}\left(\frac{x}{x_1}\right) \end{pmatrix} \begin{pmatrix} \Sigma(x_1,Q^2) \\ G(x_1,Q^2) \end{pmatrix}$$

DGLAP equations (1972-77)

$\Sigma(x,Q^2) \equiv \sum_f \left[q^f(x,Q^2) + q^{\bar{f}}(x,Q^2)\right]$ ~ flavor singlet

$\Delta^{f\bar{f}}(x,Q^2) \equiv q^f(x,Q^2) - q^{\bar{f}}(x,Q^2)$ ~ flavor non-singlet

$G(x,Q^2)$ ~ gluon distribution function (aka gluon PDF)

$\Rightarrow \alpha_s = \alpha_s(Q^2)$ as Q^2 is the only momentum scale in the equations (and the renormalization scale)

(Def.) Defining flavor singlet distribution

$$\Sigma(x, Q^2) \equiv \sum_f \left[q^f(x, Q^2) + q^{\bar{f}}(x, Q^2) \right]$$

(Def.) and flavor non-singlet

$$\Delta^{f\bar{f}}(x, Q^2) \equiv q^f(x, Q^2) - q^{\bar{f}}(x, Q^2)$$

we write

$$Q^2 \frac{\partial}{\partial Q^2} \Delta^{f\bar{f}}(x, Q^2) = \frac{\alpha(Q^2)}{2\pi} \int_x^1 \frac{dx_1}{x_1} P_{qq}\left(\frac{x}{x_1}\right) \cdot \Delta^{f\bar{f}}(x_1, Q^2)$$

and

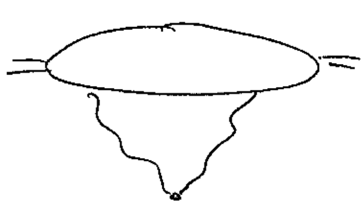
$$Q^2 \frac{\partial}{\partial Q^2} \begin{pmatrix} \Sigma(x, Q^2) \\ G(x, Q^2) \end{pmatrix} = \frac{\alpha(Q^2)}{2\pi} \int_x^1 \frac{dx_1}{x_1} \begin{pmatrix} P_{qq}\left(\frac{x}{x_1}\right) & P_{qG}\left(\frac{x}{x_1}\right) \\ P_{Gq}\left(\frac{x}{x_1}\right) & P_{GG}\left(\frac{x}{x_1}\right) \end{pmatrix} \cdot \begin{pmatrix} \Sigma(x_1, Q^2) \\ G(x_1, Q^2) \end{pmatrix}$$

Dokshitzer, Gribov, Lipatov, Altarelli, Parisi

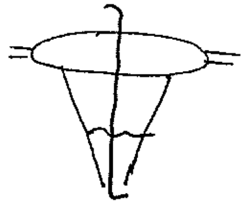
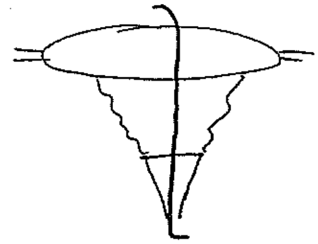
(DGLAP) Equations

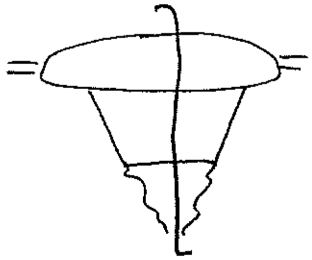
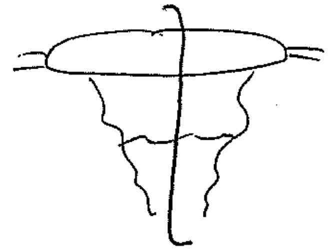
GL ~ QED case ~ '72

D, A & P ~ QCD case, '77

(Def.) $G(x, Q^2) =$  $\sim \langle A_i A_i \rangle$ in $A_+ = 0$ gauge

↖ gluon distribution function

$P_{qq} \sim$  , $P_{qG} =$ 

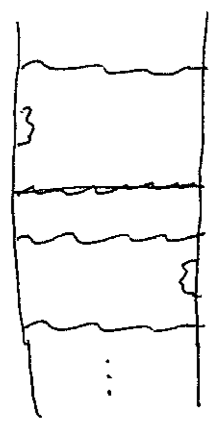
$P_{Gq} \sim$  , $P_{GG} \sim$ 

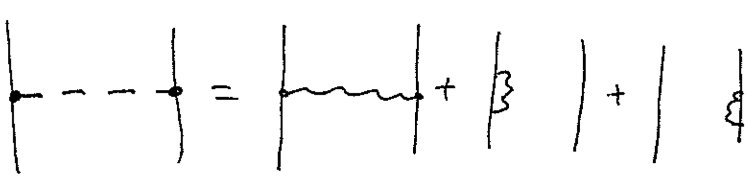
After explicit calculations one gets the <u>splitting functions</u>:

$$
\begin{cases}
P_{qq}(z) = C_F \left(\dfrac{1+z^2}{1-z} \right)_+ \\[2ex]
P_{Gq}(z) = C_F \, \dfrac{1+(1-z)^2}{z} \\[2ex]
P_{qG}(z) = N_f \left[z^2 + (1-z)^2 \right] \\[2ex]
P_{GG}(z) = 2 N_c \left[\dfrac{z}{(1-z)_+} + \dfrac{1-z}{z} + z(1-z) \right] + \dfrac{11 N_c - 2 N_f}{6} \delta(z-1)
\end{cases}
$$

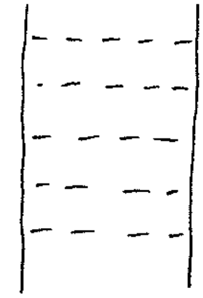
Note that $P_{Gq}(z)$ can be obtained from $P_{qq}(z)$ by substituting $z \to 1-z$ and dropping virtual corrections.

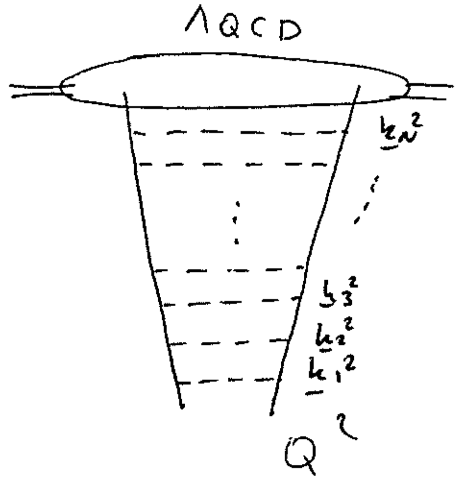
Iterate the evolution for $q_f(x, Q^2)$:



Defining 

We get a ladder diagram:



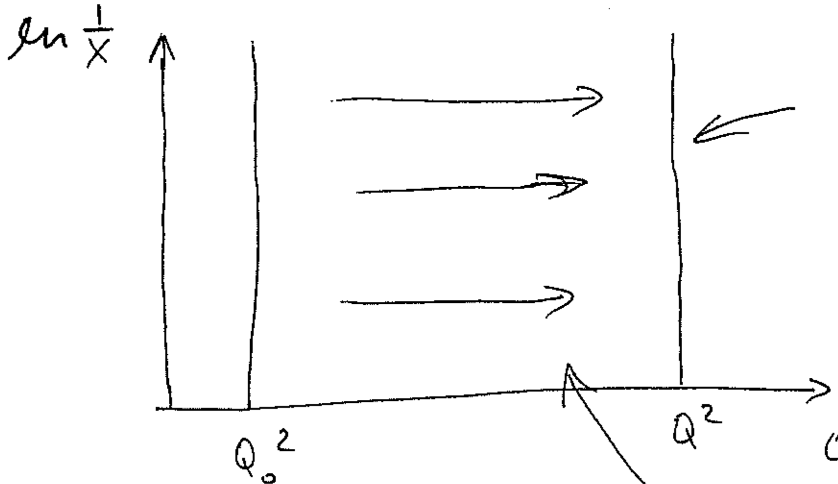


$Q^2 \gg k_1^2 \gg k_2^2 \gg \ldots \gg k_N^2 \gg \Lambda_{QCD}^2$

DGLAP resums ladder graphs with the ladder connecting scales Q^2 and Λ_{QCD}^2, $Q^2 \gg \Lambda_{QCD}^2$

such that $\ln Q^2/\Lambda_{QCD}^2 \gg 1$ and $\alpha_s \ln \frac{Q^2}{\Lambda_{QCD}^2} \sim 1$ is the resummation parameter.

How does DGLAP work?



obtain distribution at whatever Q^2 you wanted.

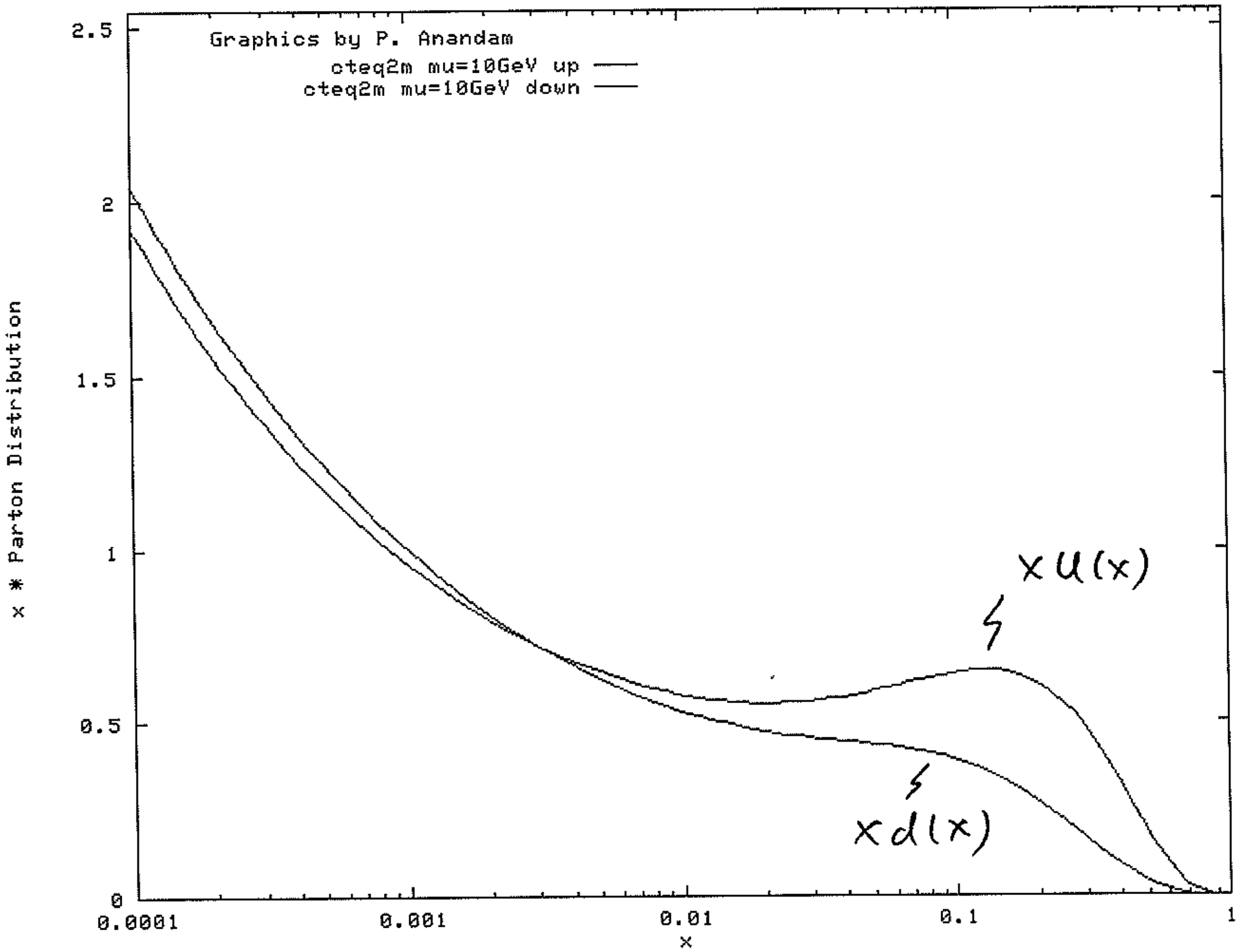
start with some initial condition = function $q^f(x, Q_0^2)$

Solve the DGLAP equations ("evolve" the distribution function)

$\Rightarrow$ people calculate PDF's (Parton Distribution Functions) & fit the data. See attachements for PDF examples.

Parton Distribution Graph

(Number of graphs plotted since 21 November 2000: 658)



$Q = 10\ GeV \Rightarrow Q^2 = 100\ GeV^2$

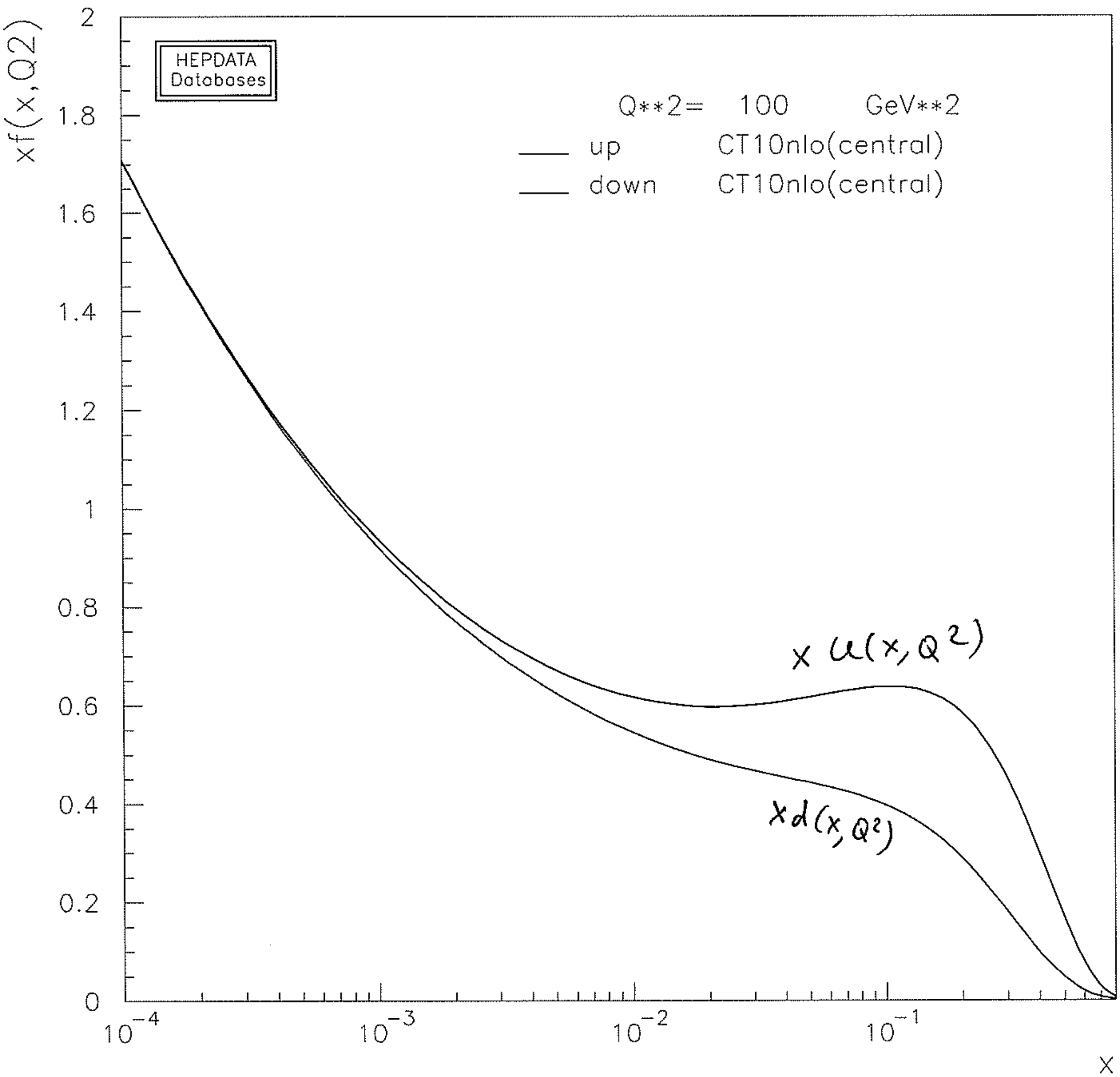
at large-x valence quarks dominate

$\Rightarrow x u_v(x) = 2\, x d_v(x)$

$\Rightarrow$ not so at small-x

(go to http://zebu.uoregon.edu/~parton/partongraph.html to plot more. ← outdated cite hepdata.cedar.ac.uk/pdf/pdf3.html ← can plot more if you wish

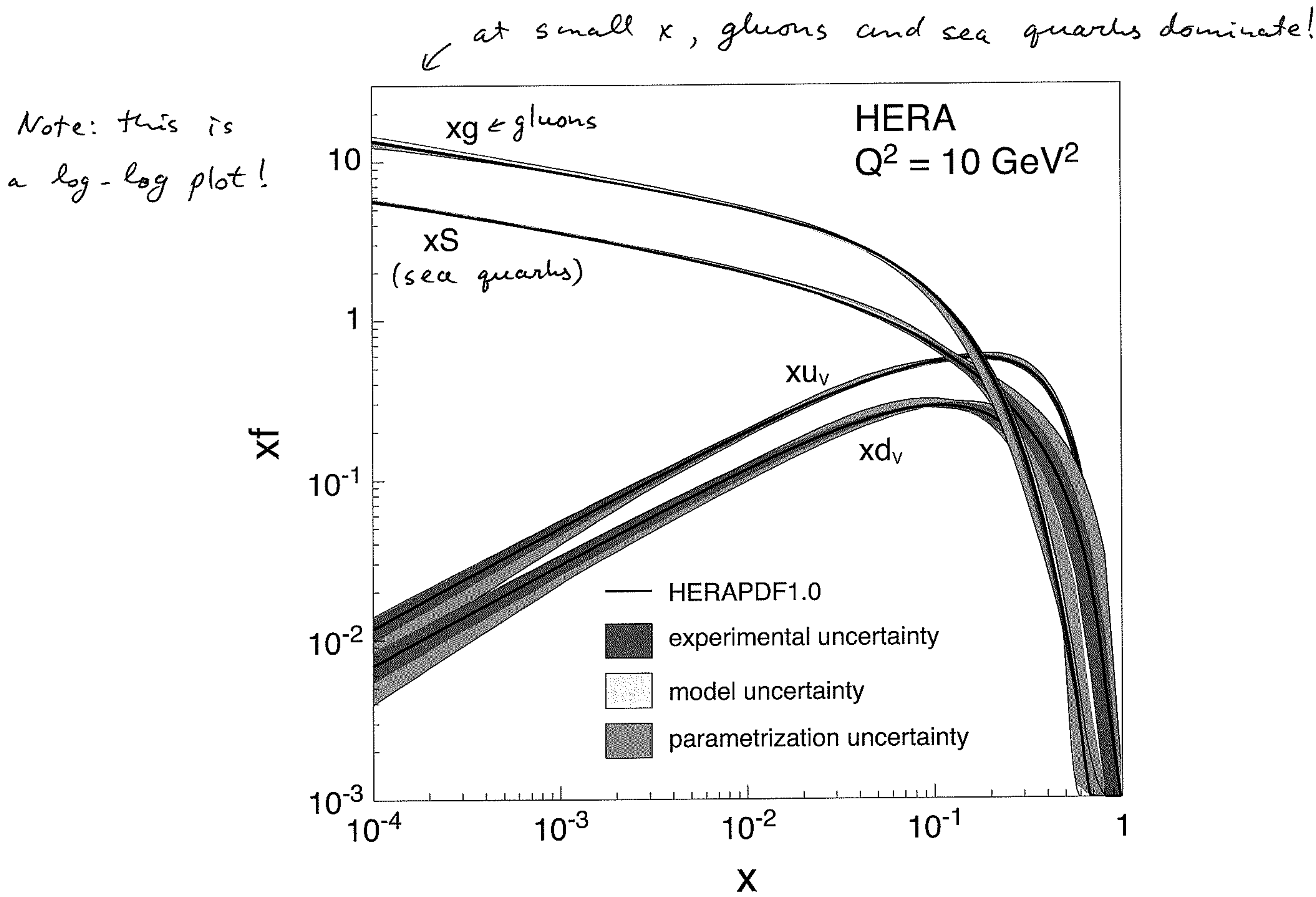
$Q^2 = 100 \ GeV^2 \quad (\Rightarrow \ Q = 10 \ GeV)$



~ at large x valence quarks dominate

$x u_v \simeq 2 \, x d_v$

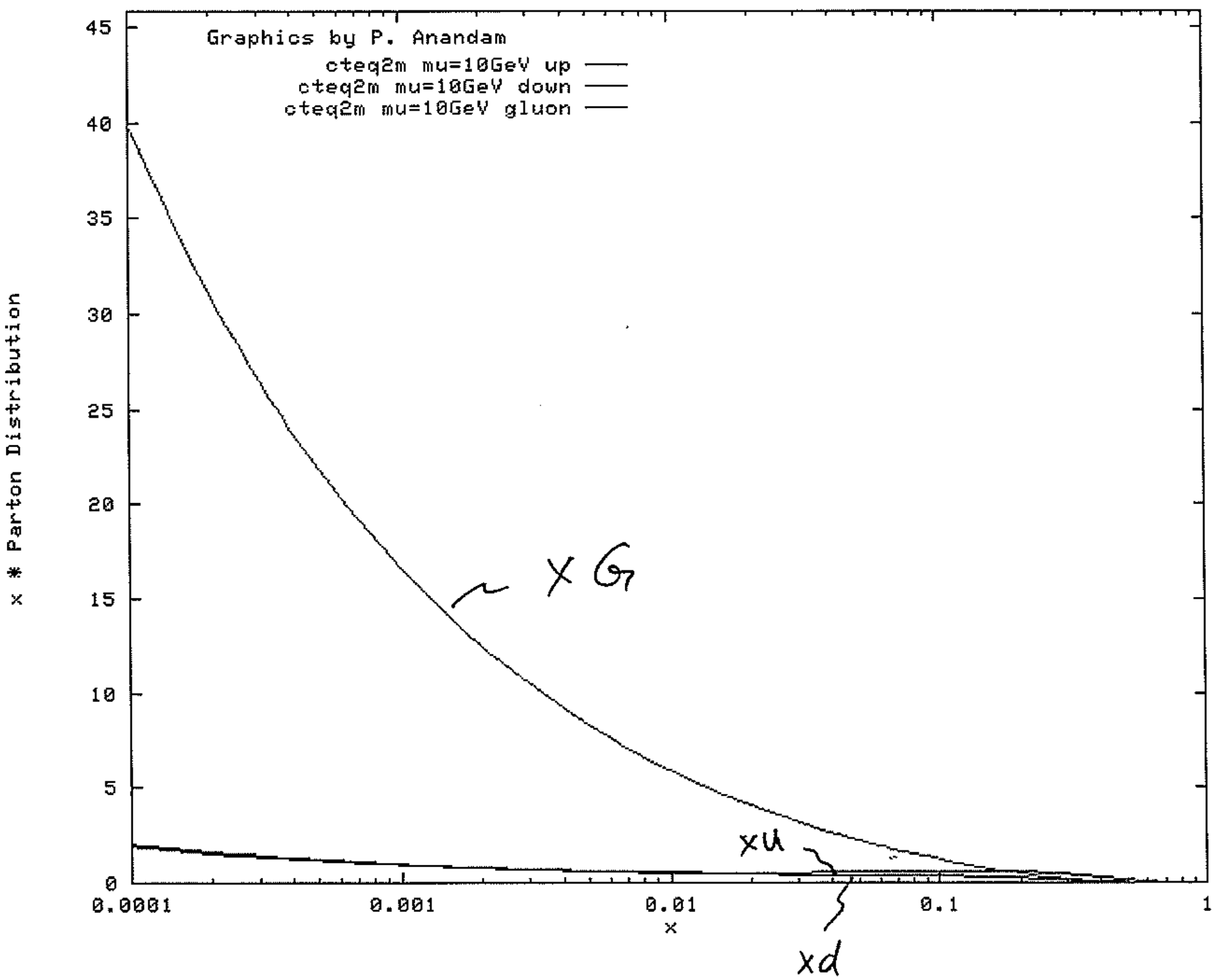
~ at small x sea quarks dominate



at small x, gluons and sea quarks dominate!
Note: this is a log-log plot!
HERA
Q2 = 10 GeV2
xg ← gluons
xS
(sea quarks)
xuv
xdv
xf
x
10
1
10-1
10-2
10-3
10-4
10-3
10-2
10-1
1
HERAPDF1.0
experimental uncertainty
model uncertainty
parametrization uncertainty

Parton Distribution Graph

(Number of graphs plotted since 21 November 2000: 659)



the same plot with xG (gluon distribution) plotted as well ....

now, who's ya daddy?

⇒ at small-x gluons dominate by far...

DGLAP at small-x.

(see attached plot)

Gluons dominate at small-x $\Rightarrow$ forget about quarks for now. Evolution for xG is

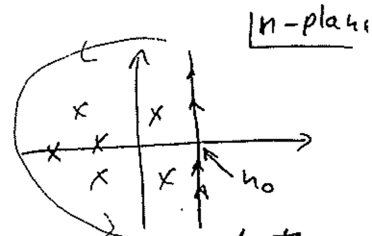
$$Q^2 \frac{\partial}{\partial Q^2} G(x, Q^2) = \frac{\alpha(Q^2)}{2\pi} \int_x^1 \frac{dx'}{x'} P_{GG}\left(\frac{x}{x'}\right) G(x', Q^2)$$

where $P_{GG}(z) = 2N_c \left[\frac{z}{(1-z)_+} + \frac{1-z}{z} + z(1-z) \right] + \frac{11 N_c - 2N_f}{6} \delta(z-1)$

$\approx \frac{2N_c}{z}$ at small z!

(Def.) Consider **moments** of $x\,G(x, Q^2)$:

$$G_n(Q^2) \equiv \int_0^1 dx\, x^{n-1} G(x, Q^2) \quad \text{(Mellin transform)}$$



to the right of all singularities

such that $G(x, Q^2) = \int \frac{dn}{2\pi i} x^{-n} G_n(Q^2)$

(Check: $\int \frac{dn}{2\pi i} x^{-n} \cdot (x')^{n-1} = \frac{1}{x'} \int \frac{dn}{2\pi i} e^{n \ln(x'/x)} = \Big| n = i\lambda + n_0$

$$= \frac{1}{x'} e^{n_0 \ln(x'/x)} \underbrace{\int_{-\infty}^{\infty} \frac{d\lambda}{2\pi} e^{i\lambda \ln(x'/x)}}_{\delta\left(\ln \frac{x'}{x}\right)} = \delta(x' - x)$$)

Multiply evolution equation for $G(x, Q^2)$ by x^{n-1} and integrate over x from $\emptyset$ to 1:

$$Q^2 \frac{\partial}{\partial Q^2} G_n(Q^2) = \frac{\alpha(Q^2)}{2\pi} \int_0^1 dx\, x^{n-1} \int_x^1 \frac{dx'}{x'} P_{GG}\left(\frac{x}{x'}\right) G(x', Q^2)$$

$$G(x,Q^2) = \int \frac{dn}{2\pi i}\, x^{-n}\, G_n(Q^2) = \int \frac{dn}{2\pi i}\, x^{-n} \cdot \int_0^1 dx' \cdot (x')^{n-1} \cdot$$

$$\cdot\, G(x',Q^2) = \int_0^1 \frac{dx'}{x'} \cdot \int \frac{dn}{2\pi i} \;\cdots\; \left(\frac{x}{x'}\right)^{-n} G(x',Q^2) =$$

$$= \int_0^1 \frac{dx'}{x'} \cdot \delta\left(\ln \frac{x}{x'}\right) G(x',Q^2) = G(x,Q^2)$$

$$G_n(Q^2) = \int_0^1 dx \cdot x^{n-1}\, G(x,Q^2) = \int_0^1 dx \cdot x^{n-1} \cdot \int \frac{dn'}{2\pi i} \cdot$$

$$\cdot\, x^{-n'}\, G_{n'}(Q^2) = \int \frac{dn'}{2\pi i}\, G_{n'}(Q^2) \cdot \int_0^1 dx \cdot x^{n-n'-1} =$$

$$= \int \frac{dn'}{2\pi i}\, G_{n'}(Q^2)\, \frac{x^{n-n'}}{n-n'}\Bigg|_0^1 = \Bigg|$$ assume $\mathrm{Re}\, n > \mathrm{Re}\, n'$

$$= \int \frac{dn'}{2\pi i}\, G_{n'}(Q^2)\, \frac{1}{n-n'} = \Bigg|$$ close the n' contour in the right half-plane

$$= G_n(Q^2).$$

$$= \frac{\alpha(Q^2)}{2\pi} \int_0^1 dx' \, (x')^{n-1} G(x', Q^2) \cdot \int_0^{\wedge} \frac{dx}{x'} \left(\frac{x}{x'}\right)^{n-1} P_{GG}\left(\frac{x}{x'}\right) = \Big| z = \frac{x}{x'}$$

$$= \frac{\alpha(Q^2)}{2\pi} \underbrace{\int_0^1 dx' \, (x')^{n-1} G(x', Q^2)}_{G_n(Q^2)} \cdot \underbrace{\int_0^1 dz \cdot z^{n-1} P_{GG}(z)}_{\gamma_{GG}^{(n)}}$$

$\gamma_{GG}^{(n)}$ ~ anomalous dimension (Def.)

$$Q^2 \frac{\partial}{\partial Q^2} G_n(Q^2) = \frac{\alpha(Q^2)}{2\pi} \gamma_{GG}^{(n)} G_n(Q^2)$$

DGLAP in Mellin space

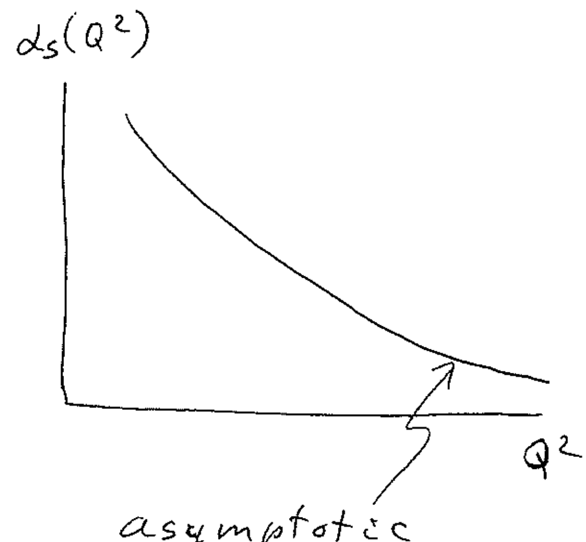
Solution:
$$G_n(Q^2) = e^{\int_{Q_0^2}^{Q^2} \frac{dQ'^2}{Q'^2} \cdot \frac{\alpha(Q'^2)}{2\pi} \gamma_{GG}^{(n)}} G_n(Q_0^2).$$

Running coupling case

$$\alpha(Q^2) = \frac{1}{\beta_2 \ln Q^2/\Lambda^2} \quad \text{with} \quad \beta_2 = \frac{11 N_c - 2 N_f}{12\pi}$$

Gross, Wilczek & Politzer

Nobel Prize of 2004



coupling is small at large Q^2 (short transverse distances $x_\perp \sim \frac{1}{Q}$) => asymptotic freedom!

$$\int_{Q_0^2}^{Q^2} \frac{dQ'^2}{Q'^2} \frac{\alpha(Q'^2)}{2\pi} \cdots = \frac{1}{2\pi\beta_2} \int_{Q_0^2}^{Q^2} \frac{dQ'^2}{Q'^2} \frac{1}{\ln Q'^2/\Lambda^2} =$$

$$= \frac{1}{2\pi\beta_2} \int_{\ln Q_0^2/\Lambda^2}^{\ln Q^2/\Lambda^2} d\ln Q'^2/\Lambda^2 \; \frac{1}{\ln Q'^2/\Lambda^2} = \frac{1}{2\pi\beta_2} \ln\left(\frac{\ln Q^2/\Lambda^2}{\ln Q_0^2/\Lambda^2}\right).$$

$$\Rightarrow \quad G_n(Q^2) = e^{\frac{\gamma_{GG}(n)}{2\pi\beta_2} \ln\left(\frac{\ln Q^2/\Lambda^2}{\ln Q_0^2/\Lambda^2}\right)} G_n(Q_0^2) \quad \Rightarrow$$

$$G(x, Q^2) = \int \frac{dn}{2\pi i} \, x^{-n} \, e^{\frac{\gamma_{GG}^{(n)}}{2\pi\beta_2} \ln\left(\frac{\ln Q^2/\Lambda^2}{\ln Q_0^2/\Lambda^2}\right)} G_n(Q_0^2).$$

at small-x: $P_{GG}(z) \simeq \frac{2N_c}{z}$

$$\Rightarrow \gamma_{GG}^{(n)} \simeq \int_0^1 dz \cdot z^{n-2} \, 2N_c = \frac{2N_c}{n-1}$$ (for $n>1$)

Evaluate the integral over n in the saddle point (a.k.a. stationary phase) approximation:

$$G(x, Q^2) = \int \frac{dn}{2\pi i} \, e^{n \ln\frac{1}{x} + \frac{N_c}{n-1} \frac{1}{\pi\beta_2} \ln\left(\frac{\ln(Q^2/\Lambda^2)}{\ln(Q_0^2/\Lambda^2)}\right)} G_n(Q_0^2)$$

Assume that most n-dependence is in the exponent. At small-x $\ln\frac{1}{x}$ is very large $\Rightarrow$

$\Rightarrow$ the exponent oscillates wildly as n varies.

Oscillations are not there only at the saddle point:

$$\frac{d}{dn}\left[n \ln\frac{1}{x} + \frac{N_c}{n-1}\,\frac{1}{\pi\beta_2}\,\ln\left(\frac{\ln Q^2/\Lambda^2}{\ln Q_0^2/\Lambda^2}\right)\right]\Bigg|_{n=n_0} = 0$$

$$\ln\frac{1}{x} - \frac{N_c}{(n_0-1)^2}\,\frac{1}{\pi\beta_2}\,\ln\left(\frac{\ln(Q^2/\Lambda^2)}{\ln(Q_0^2/\Lambda^2)}\right) = 0$$

$$n_0 - 1 = \pm\sqrt{\frac{N_c}{\pi\beta_2}\ln\left(\frac{\ln(Q^2/\Lambda^2)}{\ln(Q_0^2/\Lambda^2)}\right)\frac{1}{\ln\frac{1}{x}}}$$

"+" dominates (gives larger contribution).
to $(n_0-1)\ln\frac{1}{x}$

To estimate the integral we define the power of the exponent

$$P(n) = n\ln\frac{1}{x} + \frac{N_c}{n-1}\,\frac{1}{\pi\beta_2}\,\ln\left(\frac{\ln(Q^2/\Lambda^2)}{\ln(Q_0^2/\Lambda^2)}\right)$$

and expand

$$P(n) \approx P(n_0) + \frac{1}{2}(n-n_0)^2\,P''(n_0)$$

where
$$P''(n_0) = +\frac{2N_c}{(n_0-1)^3}\,\frac{1}{\pi\beta_2}\,\ln\frac{\ln Q^2/\Lambda^2}{\ln Q_0^2/\Lambda^2} = \frac{2N_c}{\pi\beta_2}\ln\frac{\ln Q^2/\Lambda^2}{\ln Q_0^2/\Lambda^2}\,.$$

$$\left(\frac{\pi b}{N_c}\right)^{3/2}\left[\ln\left(\frac{\ln Q^2/\Lambda^2}{\ln Q_0^2/\Lambda^2}\right)\right]^{-3/2}\ln^{3/2}\frac{1}{x} = 2\left(\frac{\pi\beta_2}{N_c}\right)^{1/2}\ln^{3/2}\frac{1}{x}\cdot\left[\ln\frac{\ln Q^2/\Lambda^2}{\ln Q_0^2/\Lambda^2}\right]^{-1/2}.$$

$$P(n_0) = \ln\frac{1}{x} + 2\sqrt{\frac{N_c}{\pi\beta_2}\ln\left(\frac{\ln Q^2/\Lambda^2}{\ln Q_0^2/\Lambda^2}\right)\ln\frac{1}{x}}\,.$$

$$\int \frac{dn}{2\pi i}\, e^{P(n_0)+\frac{1}{2}(n-n_0)^2 P''(n_0)} = \Big| n-n_0 = i\xi = \int_{-\infty}^{\infty} \frac{d\xi}{2\pi}\, e^{P(n_0)-\frac{1}{2}\xi^2 P''(n_0)} =$$

$$= \frac{1}{2\pi}\, e^{P(n_0)} \sqrt{\frac{2\pi}{P''(n_0)}} = \frac{e^{P(n_0)}}{\sqrt{2\pi P''(n_0)}}$$

we obtain

$$xG(x,Q^2) = G_{n_0}(Q_0^2)\cdot e^{2\sqrt{\frac{N_c}{\pi\beta_2}\ln\left(\frac{\ln Q^2/\Lambda^2}{\ln Q_0^2/\Lambda^2}\right)\ln\frac{1}{x}}}\cdot\frac{1}{\sqrt{4\pi}}$$

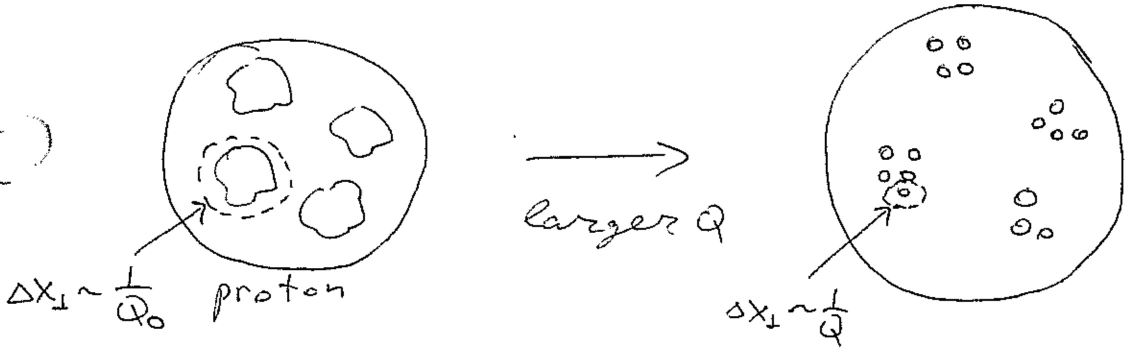
$$\cdot\left(\frac{N_c}{\pi\beta_2}\right)^{1/4}\ln^{-3/4}\frac{1}{x}\cdot\left[\ln\left(\frac{\ln Q^2/\Lambda^2}{\ln Q_0^2/\Lambda^2}\right)\right]^{1/4}$$

also note that xG grows with Q^2

Therefore, $xG \sim e^{2\sqrt{\frac{N_c}{\pi\beta_2}\ln\frac{1}{x}\ln\left(\frac{\ln Q^2/\Lambda^2}{\ln Q_0^2/\Lambda^2}\right)}}$

xG grows at small-x, slower than a power of x but faster than any power of $\ln\frac{1}{x}$. $\Rightarrow$ may explain rise of xG at small-x...

How DGLAP works: we increase Q/resolution, see more partons



Renormalization Groups.

A Note on the Saddle Point Method

(aka the Method of Steepest Descent)

$$I(\lambda) = \int_C dz\, g(z)\, e^{\lambda f(z)}$$

$f(z)$, $g(z)$ analytic functions

$\lambda \gg 1$ ~ large parameter

(i) Find a point z_0 such that $\boxed{f'(z=z_0)=0}$.

(ii) Deform the contour C to go through z_0 along the $\text{Im}\, f(z) = \text{Im}\, f(z_0)$ line. (Line of steepest descent.)

(iii) Evaluate the resulting integral. In most practical applications one can approximate $f(z) \simeq f(z_0) + \frac{1}{2} f''(z_0)(z-z_0)^2$ such that

$$I \simeq g(z_0)\, e^{\lambda f(z_0)} \int dz\, e^{\frac{\lambda}{2} f''(z_0)(z-z_0)^2}$$

for $\lambda \gg 1$.