Axial Anomaly

Consider massless QED as an example:

$$\mathcal{L} = \bar{\psi}\, i\gamma\cdot\partial\psi - e\bar{\psi}\gamma^\mu A_\mu \psi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu}$$

$\psi \sim$ electron field, $A_\mu \sim$ photon field.

$\mathcal{L}$ is invariant under the following *global* symmetries:

(i) $\psi \to e^{i\alpha}\psi \Rightarrow \bar{\psi} \to e^{-i\alpha}\bar{\psi} \Rightarrow \mathcal{L}$ is invariant

under $U(1)$ global symmetry.

The corresponding current is $j_\mu = -ie\bar{\psi}\gamma^\mu\psi$.

It is conserved: $\partial_\mu j^\mu = 0$

as $\gamma_5^\dagger = \gamma_5$

(ii) $\psi \to e^{i\gamma_5\alpha}\psi \Rightarrow \bar{\psi} = \psi^\dagger\gamma^0 \to \psi^\dagger e^{-i\gamma_5\alpha}\gamma^0$

$\{\gamma_5,\gamma^0\}=0$

$= \psi^\dagger\gamma^0 e^{i\gamma_5\alpha} = \bar{\psi}e^{i\alpha\gamma_5} \Rightarrow$

$$\bar{\psi}\, i\gamma^\mu D_\mu\psi \to \bar{\psi}e^{i\alpha\gamma_5} i\gamma^\mu D_\mu e^{i\alpha\gamma_5}\psi =$$

$$= \bar{\psi}e^{i\alpha\gamma_5}e^{-i\alpha\gamma_5} i\gamma^\mu D_\mu\psi = \bar{\psi}\, i\gamma^\mu D_\mu\psi$$

as $\{\gamma_5,\gamma^\mu\}=0$

$\Rightarrow$ corresponding conserved current is

(massless fermions)

$j_\mu^5 = \bar{\psi}\gamma^\mu\gamma_5\psi$: $\partial_\mu j^{5\mu} = 0$

$\Rightarrow$ seems like massless QED Lagrangian is invariant under the axial symmetry $U_A(1)$

$\Rightarrow \mathcal{L}_{QED}$ is $U(1) \otimes U_A(1)$ invariant.

However, this is <u>not true</u> when quantum corrections are included. $\Rightarrow$ we will see that $\partial_\mu j^{5\mu} \neq 0$ if quantum corrections are counted.

Consider $\partial_\mu j^{5\mu} = \partial_\mu (\bar{\psi} \gamma^\mu \gamma_5 \psi)$ $\Rightarrow$ in momentum space $\partial_\mu \to -i k_\mu$, the vertex has $\gamma^\mu \gamma_5$.

Consider 3-point correlator:

$$T_{\mu\nu\rho}(k_1, k_2) = i \int d^4x_1 \, d^4x_2 \, e^{ik_1\cdot x_1 + ik_2 \cdot x_2} \langle 0 | T\big(j_\mu(x_1) \cdot j_\nu(x_2) \cdot j^5_\rho(0)\big) | 0 \rangle$$

~~Writing $T_{\mu\nu\rho}(k_1,k_2) = i\int d^4x_1 \, d^4x_2 \, e^{i(x_1-y)k_1 + ik_2(x_2-y)}$~~

~~$\langle 0 | T[j_\mu(x_1) j_\nu(x_2) j^5_\rho(y)] | 0 \rangle$ $\Rightarrow$ we expect~~

~~$(k_1+k_2)^\rho T_{\mu\nu\rho}(k_1,k_2) = i\int d^4x_1 \, d^4x_2 \, i \partial^\rho_y \left[e^{ik_1(x_1-y)+ik_2(x_2-y)} \right]$~~

~~$\langle 0 | T[j_\mu(x_1) j_\nu(x_2) j^5_\rho(y)] | 0 \rangle$ = (parts)~~

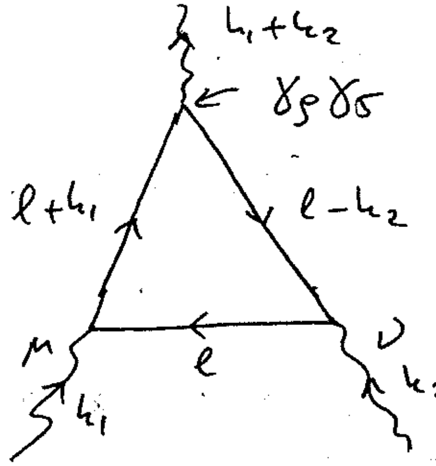
One can show that $\partial_\mu j^{5\mu} = 0$ would lead to

$$(k_1+k_2)^\rho \, T_{\mu\nu\rho}(k_1, k_2) = 0 .$$

Check this statement:

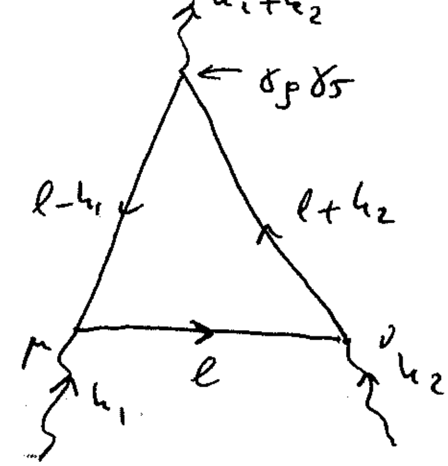
$T_{\mu\nu\rho}(k_1,k_2) =$

arrow indicates both momentum & fermion #.



graph A

$+$



graph B

((Can write $T_{\mu\nu\rho}(k_1,k_2) \underset{\substack{x_1\to 0\\ 0\to x_1}}{=} i\int d^4x_1\, d^4x_2\, e^{-ik_1\cdot x_1 + ik_2\cdot(x_2-x_1)}$

$\cdot\langle 0|T[j_\mu(0)\, j_\nu(x_2)\, j_\rho^5(x_1)]|0\rangle \Rightarrow (k_1+k_2)^\rho\, T_{\mu\nu\rho} =$

$= i\int d^4x_1\, d^4x_2\;\; i\,\partial^\rho_{x_1}\left(e^{ik_2\cdot x_2 - i(k_1+k_2)\cdot x_1}\right)\langle 0|T[j_\mu(0)j_\nu(x_2)j_\rho^5(x_1)]|0\rangle$

$\Rightarrow$ (parts) $= \int d^4x_1\, d^4x_2\, e^{ik_2\cdot x_2 - i(k_1+k_2)\cdot x_1}\langle 0|T[j_\mu(0)j_\nu(x_2)\cdot$

+ more work (equal-time commut. relations, etc) ~ see attached pages

$\cdot\,\partial^\rho j_\rho^5(x_1)]|0\rangle = 0$ if $\partial^\rho j_\rho^5 = 0$.)

fermion loop

$$-i\,T_{\mu\nu\rho} = -(-ie)^2\int\frac{d^4l}{(2\pi)^4}\,\mathrm{Tr}\left[\gamma_\rho\gamma_5\,\frac{i}{\not l+\not k_1}\,\gamma_\mu\,\frac{i}{\not l}\,\gamma_\nu\,\frac{i}{\not l-\not k_2}\right]$$

$$-(-ie)^2\int\frac{d^4l}{(2\pi)^4}\,\mathrm{Tr}\left[\gamma_\rho\gamma_5\,\frac{i}{\not l+\not k_2}\,\gamma_\nu\,\frac{i}{\not l}\,\gamma_\mu\,\frac{i}{\not l-\not k_1}\right] =$$

$$= -ie^2\int\frac{d^4l}{(2\pi)^4}\,\frac{\mathrm{Tr}[\gamma_\rho\gamma_5(\not l+\not k_1)\gamma_\mu\not l\gamma_\nu(\not l-\not k_2)]}{(l^2+i\varepsilon)((l+k_1)^2+i\varepsilon)((l-k_2)^2+i\varepsilon)} -$$

$$-ie^2\int\frac{d^4\ell}{(2\pi)^4}\;\frac{Tr[\gamma_\rho\gamma_5(\not\ell+\not k_2)\gamma_\nu\not\ell\gamma_\mu(\not\ell-\not k_1)]}{(\ell^2+i\varepsilon)((\ell+k_2)^2+i\varepsilon)((\ell-k_1)^2+i\varepsilon)}$$ (118.)

$$\Rightarrow (k_1+k_2)^\rho\, T_{\mu\nu\rho} = e^2\int\frac{d^4\ell}{(2\pi)^4}\left\{\underbrace{\frac{Tr[(\not k_1+\not k_2)\gamma_5(\not\ell+\not k_1)\gamma_\mu\not\ell\gamma_\nu(\not\ell-\not k_2)]}{(\ell^2+i\varepsilon)((\ell+k_1)^2+i\varepsilon)((\ell-k_2)^2+i\varepsilon)}}_{=A}\right.$$

$$\left.+\underbrace{\frac{Tr[(\not k_1+\not k_2)\gamma_5(\not\ell+\not k_2)\gamma_\nu\not\ell\gamma_\mu(\not\ell-\not k_1)]}{(\ell^2+i\varepsilon)((\ell+k_2)^2+i\varepsilon)((\ell-k_1)^2+i\varepsilon)}}_{=B}\right\}$$

Numerator of A $= Tr[(\not k_1+\not\ell-(\not\ell-\not k_2))\gamma_5(\not\ell+\not k_1)\gamma_\mu\not\ell$

$\gamma_\nu(\not\ell-\not k_2)] = -(\ell+k_1)^2\, Tr[\gamma_5\gamma_\mu\not\ell\gamma_\nu(\not\ell-\not k_2)] -$

$-(\ell-k_2)^2\, Tr[\gamma_5(\not\ell+\not k_1)\gamma_\mu\not\ell\gamma_\nu]$

Numerator of B $= Tr[((\not k_2+\not\ell)-(\not\ell-\not k_1))\gamma_5(\not\ell+\not k_2)\cdot$

$\cdot\gamma_\nu\not\ell\gamma_\mu(\not\ell-\not k_1)] = -(\ell+k_2)^2\, Tr[\gamma_5\gamma_\nu\not\ell\gamma_\mu(\not\ell-\not k_1)]$

$-(\ell-k_1)^2\, Tr[\gamma_5(\not\ell+\not k_2)\gamma_\nu\not\ell\gamma_\mu] \Rightarrow$

$$(k_1+k_2)^\rho\, T_{\mu\nu\rho} = -e^2\int\frac{d^4\ell}{(2\pi)^4}\frac{1}{\ell^2+i\varepsilon}\left\{\overset{①}{\frac{Tr[\gamma_5\gamma_\mu\not\ell\gamma_\nu(\not\ell-\not k_2)]}{(\ell-k_2)^2+i\varepsilon}}+\right.$$

$$+\overset{②}{\frac{Tr[\gamma_5(\not\ell+\not k_1)\gamma_\mu\not\ell\gamma_\nu]}{(\ell+k_1)^2+i\varepsilon}}+\overset{③}{\frac{Tr[\gamma_5\gamma_\nu\not\ell\gamma_\mu(\not\ell-\not k_1)]}{(\ell-k_1)^2+i\varepsilon}}+$$

$$\left.+\overset{④}{\frac{Tr[\gamma_5(\not\ell+\not k_2)\gamma_\nu\not\ell\gamma_\mu]}{(\ell+k_2)^2+i\varepsilon}}\right\}$$

$$T_{\mu\nu\rho}(k_1,k_2) = i\int d^4x_1\, d^4x_2\, e^{-ik_1\cdot x_1 + ik_2\cdot(x_2-x_1)} \cdot$$

$$\cdot \langle 0|T\{j_\mu(0)\, j_\nu(x_2)\, j_\rho^5(x_1)\}|0\rangle \quad \Rightarrow$$

$$(k_1+k_2)^\rho T_{\mu\nu\rho} = i\int d^4x_1\, d^4x_2 \;\; i\,\partial^\rho_{x_1}\left(e^{ik_2\cdot x_2 - i(k_1+k_2)\cdot x_1}\right) \cdot$$

$$\cdot \langle 0|T\{j_\mu(0)\, j_\nu(x_2)\, j_\rho^5(x_1)\}|0\rangle = \Big|\text{parts}$$

$$= \int d^4x_1\, d^4x_2\; e^{ik_2\cdot x_2 - i(k_1+k_2)\cdot x_1} \langle 0|\,\partial^\rho_{x_1} \cdot T\{j_\mu(0) \cdot$$

$$\cdot j_\nu(x_2)\, j_\rho^5(x_1)\}|0\rangle .$$

Now, $T\{j_\mu(0)\, j_\nu(x_2)\, j_\rho^5(x_1)\} = \theta(-x_2^0)\,\theta(x_2^0-x_1^0) \cdot$

$$\cdot j_\mu(0)\, j_\nu(x_2)\, j_\rho^5(x_1) + \theta(-x_1^0)\,\theta(x_1^0-x_2^0)\, j_\mu(0)\, j_\rho^5(x_1)\, j_\nu(x_2)$$

$$+\, \theta(x_2^0)\,\theta(-x_1^0)\, j_\nu(x_2)\, j_\mu(0)\, j_\rho^5(x_1) + \theta(x_2^0-x_1^0)\,\theta(x_1^0)\, j_\nu(x_2)\, j_\rho^5(x_1)\, j_\mu(0)$$

$$+\, \theta(x_1^0)\,\theta(-x_2^0)\, j_\rho^5(x_1)\, j_\mu(0)\, j_\nu(x_2) + \theta(x_1^0-x_2^0)\,\theta(x_2^0)\, j_\rho^5(x_1)\, j_\nu(x_2)\, j_\mu(0)$$

$$\Rightarrow \partial^\rho_{x_1} T\{j_\mu(0)\, j_\nu(x_2)\, j_\rho^5(x_1)\} = T\{j_\mu(0)\, j_\nu(x_2)\, \overbrace{\partial^\rho_{x_1} j_\rho^5(x_1)}^{=0 \text{ if conserved}}\} +$$

$$+\, \delta^{\rho 0}\Big[-\theta(-t_2)\,\delta(t_1-t_2)\, j_\mu(0)\, j_\nu(x_2)\, j_\rho^5(x_1) - \delta(t_1)\,\theta(-t_2)\, j_\mu(0) \cdot$$

$$\cdot j_\rho^5(x_1)\, j_\nu(x_2) + \theta(-t_2)\,\delta(t_1-t_2)\, j_\mu(0)\, j_\rho^5(x_1)\, j_\nu(x_2)$$

(117'')

$$- \theta(t_2)\,\delta(t_1)\, j_\nu(x_2)\, j_\mu(0)\, j_\rho^5(x_1) - \delta(t_1-t_2)\,\theta(t_2)\, j_\nu(x_2) j_\rho^5(x_1) j_\mu(0)$$

$$+ \theta(t_2)\,\delta(t_1)\, j_\nu(x_2)\, j_\rho^5(x_1)\, j_\mu(0) + \delta(t_1)\,\theta(-t_2)\, j_\rho^5(x_1)\, j_\mu(0)\, j_\nu(x_2)$$

$$+ \delta(t_1-t_2)\,\theta(t_2)\, j_\rho^5(x_1)\, j_\nu(x_2)\, j_\mu(0) \Big] = \delta^{\rho 0} \cdot \Big[-\theta(-t_2)\,\delta(t_1-t_2)$$

$$\cdot j_\mu(0)\,[j_\nu(x_2), j_\rho^5(x_1)] - \theta(t_2)\,\delta(t_1-t_2)\,[j_\nu(x_2), j_\rho^5(x_1)]\, j_\mu(0)$$

$$- \theta(-t_2)\,\delta(t_1)\,[j_\mu(0), j_\rho^5(x_1)]\, j_\nu(x_2) - \theta(t_2)\,\delta(t_1) \cdot j_\nu(x_2)$$

$$\cdot [j_\mu(0), j_\rho^5(x_1)] \Big] = 0$$

The expression is zero because all the equal-time correlation relations are zero for $\rho = 0$. For instance,

$$\delta^{\rho 0}\,\delta(t_1-t_2)\,[j_\nu(x_2), j_\rho^5(x_1)] \underset{\uparrow}{=} \delta^4(x_1-x_2)\,\psi^+\,[\gamma^0\gamma^\nu, \gamma^0\gamma^0\gamma^5]\,\psi$$

problem 3, HW5, part a

$$= \delta^4(x_1-x_2)\,\psi^+\,[\gamma^0\gamma^\nu, \gamma^5]\,\psi = 0 \quad \text{as } [\gamma^0\gamma^\nu, \gamma^5] = 0.$$

(119')

Shifts in ill-defined integrals:

$$\int_0^\infty dx = \Big|\, x \to x + a = y \,= \int_a^\infty dy = -\int_0^a dy + \int_0^\infty dy = -a + \int_0^\infty dy$$

$\Rightarrow a = 0$ for $\forall$ real a $\Rightarrow$ all #'s are zero!

Now, if in (4) we shift $\ell \to \ell - k_2$ $\Rightarrow$ it would cancel (1) as (1) + (4) $\propto \{\gamma_5, \gamma_\mu\} = 0$.

In (3) shift $\ell \to \ell + k_1$ $\Rightarrow$ cancel (2).

$\Rightarrow$ seems to get $(k_1 + k_2)^\rho \, T_{\mu\nu\rho} = 0$ in expectation with $\partial^\rho j^5_\rho = 0$ ...

<u>Problem</u> at large-ℓ all integrals are quadratically divergent!

We get (1) ~ (2) ~ (3) ~ (4) ~ $\int d^4\ell \, \frac{1}{\ell^2} \sim \int d\ell \cdot \ell \sim \infty^2$.

$\Rightarrow$ can't shift variables in divergent integrals!

$$\int_0^\infty d\ell \cdot \ell \xrightarrow[\ell \to \ell + a]{\text{shift}} \int_{-a}^\infty d\ell \cdot (\ell + a) = \int_0^\infty d\ell \cdot (\ell + a) + \int_{-a}^0 d\ell \cdot (\ell + a)$$

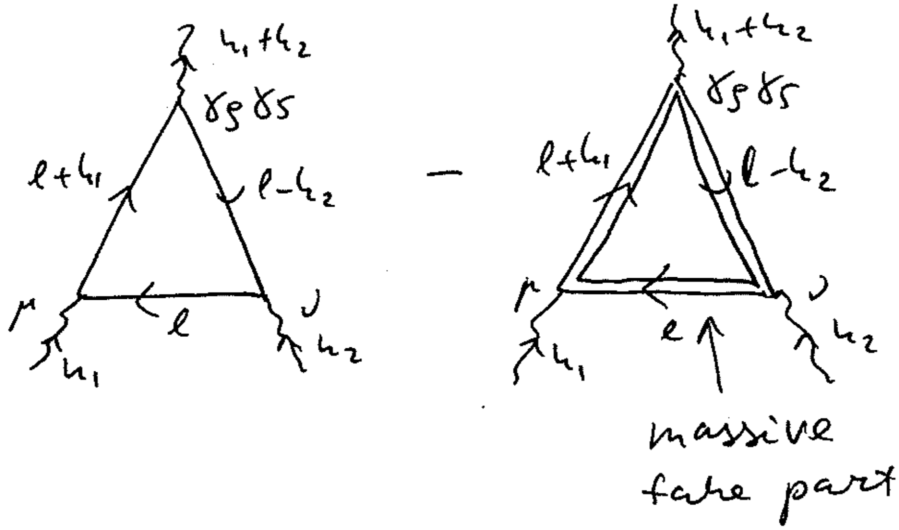
$$= \int_0^\infty d\ell \cdot \ell + a \cdot \int_0^\infty d\ell + \left(\frac{\ell^2}{2} + a\ell\right)\Bigg|_{-a}^0 = \underbrace{\int_0^\infty d\ell \cdot \ell}_{\text{old integral}} + a \underbrace{\int_0^\infty d\ell}_{\infty} + \frac{a^2}{2}$$

$\Rightarrow$ did not survive the shift, got corrections?

$\Rightarrow$ ill-defined procedure $\Rightarrow$ need to make integrals finite, need to regulate them!

We'll use Pauli-Villars regularization: introduce (subtract) a new particle with mass m, which is then taken to ∞ to eliminate the particle.

$T_{\mu\nu\rho} =$



$$+ \underbrace{\begin{pmatrix} k_1 \leftrightarrow k_2 \\ \mu \leftrightarrow \nu \end{pmatrix}}_{\text{graph B}}$$

$$T_{\mu\nu\rho} = -e^2 \int \frac{d^4\ell}{(2\pi)^4} \left\{ \frac{\mathrm{Tr}\left[\gamma_\rho\gamma_5 (\not\ell + \not k_1)\gamma_\mu \not\ell \gamma_\nu (\not\ell - \not k_2)\right]}{(\ell^2+i\varepsilon)((\ell+k_1)^2+i\varepsilon)((\ell-k_2)^2+i\varepsilon)} \right.$$

$$\left. - \frac{\mathrm{Tr}\left[\gamma_\rho\gamma_5(\not\ell+\not k_1+m)\gamma_\mu(\not\ell+m)\gamma_\nu(\not\ell-\not k_2+m)\right]}{(\ell^2-m^2+i\varepsilon)((\ell+k_1)^2-m^2+i\varepsilon)((\ell-k_2)^2-m^2+i\varepsilon)} \right\} + \begin{pmatrix} k_1\leftrightarrow k_2 \\ \mu\leftrightarrow\nu \end{pmatrix}$$

The second Tr has only even powers of m in its expansion (m^0 & m^2). (Tr of odd # of γ's is zero.) Write:

$$(k_1+k_2)^\rho\, T_{\mu\nu\rho} = -e^2 \int \frac{d^4\ell}{(2\pi)^4} \left\{ \mathrm{Tr}\left[(\not k_1+\not k_2)\gamma_5(\not\ell+\not k_1)\gamma_\mu \not\ell \gamma_\nu \cdot \right.\right.$$

$$\left. \cdot (\not\ell - \not k_2)\right] \left[\frac{1}{\ell^2(\ell+k_1)^2(\ell-k_2)^2} - \frac{1}{(\ell^2-m^2)\left[(\ell+k_1)^2-m^2\right]\left[(\ell-k_2)^2-m^2\right]} \right] -$$

$$\left. - \frac{m^2\text{-term in 2nd trace } (O(\ell))}{[\ell^2-m^2][(\ell+k_1)^2-m^2][(\ell-k_2)^2-m^2]} \right\} + \begin{pmatrix} k_1\leftrightarrow k_2 \\ \mu\leftrightarrow\nu \end{pmatrix}$$

Now the integral is convergent & shifts are allowed!

$\Rightarrow$ the $m=0$ (first) term in [...] vanishes like before.

For the term in [...] containing m^2 write:

$$\mathrm{Tr}\left[\left(\not{k}_1+\not{l}-(\not{l}-\not{k}_2)\right)\gamma_5(\not{l}+\not{k}_1)\gamma_\mu\not{l}\gamma_\nu(\not{l}-\not{k}_2)\right]=$$

$$=-(l+k_1)^2\,\mathrm{Tr}\left[\gamma_5\gamma_\mu\not{l}\gamma_\nu(\not{l}-\not{k}_2)\right]-(l-k_2)^2\,\mathrm{Tr}\left[\gamma_5(\not{l}+\not{k}_1)\cdot\gamma_\mu\not{l}\gamma_\nu\right]=-\left[(l+k_1)^2-m^2\right]\mathrm{Tr}\left[\gamma_5\gamma_\mu\not{l}\gamma_\nu(\not{l}-\not{k}_2)\right]$$

$$-\left[(l-k_2)^2-m^2\right]\mathrm{Tr}\left[\gamma_5(\not{l}+\not{k}_1)\gamma_\mu\not{l}\gamma_\nu\right]-m^2\left(\mathrm{Tr}\left[\gamma_5\gamma_\mu\not{l}\gamma_\nu\cdot(\not{l}-\not{k}_2)\right]+\mathrm{Tr}\left[\gamma_5(\not{l}+\not{k}_1)\gamma_\mu\not{l}\gamma_\nu\right]\right)$$

First two terms also cancel after shifts.

We get:

$$(k_1+k_2)^\rho\,T_{\mu\nu\rho}=-e^2\int\frac{d^4l}{(2\pi)^4}\,m^2\left\{\frac{1}{\left[l^2-m^2\right]\left[(l+k_1)^2-m^2\right]\left[(l-k_2)^2-m^2\right]}\right.$$

$$\cdot\left\{\mathrm{Tr}\left[\gamma_5\gamma_\mu\not{l}\gamma_\nu(\not{l}-\not{k}_2)\right]+\mathrm{Tr}\left[\gamma_5(\not{l}+\not{k}_1)\gamma_\mu\not{l}\gamma_\nu\right]-\right.$$

$$\left.-\mathrm{Tr}\left[(\not{k}_1+\not{k}_2)\gamma_5\gamma_\mu\gamma_\nu(\not{l}-\not{k}_2)\right]-\mathrm{Tr}\left[(\not{k}_1+\not{k}_2)\gamma_5\cdot(\not{l}+\not{k}_1)\gamma_\mu\gamma_\nu\right]-\mathrm{Tr}\left[(\not{k}_1+\not{k}_2)\gamma_5\gamma_\mu\not{l}\gamma_\nu\right]\right\}+\begin{pmatrix}k_1\leftrightarrow k_2\\ \mu\leftrightarrow\nu\end{pmatrix}$$

m^2 terms in Tr.

$$=\left(\text{as } \mathrm{Tr}\left[\gamma^5\gamma^\mu\gamma^\nu\gamma^\alpha\gamma^\beta\right]=-4i\,\varepsilon^{\mu\nu\alpha\beta}\right)=$$

$$= 4ie^2\,\varepsilon^{\mu\nu\alpha\beta}\int\frac{d^4\ell}{(2\pi)^4}\,\frac{m^2}{(\ell^2-m^2)\left[(\ell+k_1)^2-m^2\right]\left[(\ell-k_2)^2-m^2\right]}$$

$$\cdot\Big\{-\ell_\alpha(\cancel{\ell}-k_2)_\beta+\ell_\alpha(\cancel{\ell}+k_1)_\beta-(\ell-k_2)_\alpha(k_1+k_2)_\beta+(k_1+k_2)_\alpha(\ell+k_1)_\beta$$

$$-(k_1+k_2)_\alpha\,\ell_\beta\Big\}+\begin{pmatrix}k_1\leftrightarrow k_2\\ \mu\leftrightarrow\nu\end{pmatrix}=4ie^2\,\varepsilon^{\mu\nu\alpha\beta}\int\frac{d^4\ell}{(2\pi)^4}\cdot$$

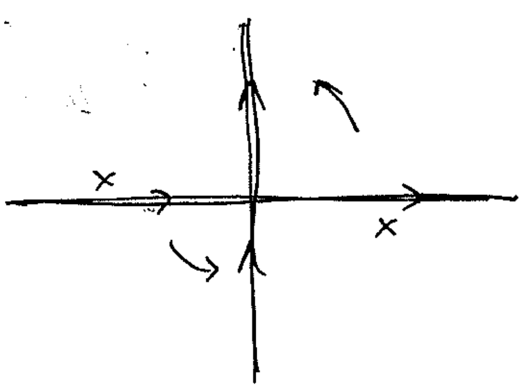
$$\frac{m^2}{\left[\ell^2-m^2\right]\left[(\ell+k_1)^2-m^2\right]\left[(\ell-k_2)^2-m^2\right]}\Big\{\cancel{\ell_\alpha k_{2\beta}}+\cancel{\ell_\alpha k_{1\beta}}-$$

$$-\cancel{\ell_\alpha(k_1+k_2)_\beta}+\cancel{(k_1+k_2)_\alpha\,\ell_\beta}-\cancel{(k_1+k_2)_\alpha\,\ell_\beta}+k_{2\alpha}(k_1+\underset{0}{\cancel{k_2}})_\beta$$

$$+k_{1\beta}(\underset{0}{\cancel{k_1}}+k_2)_\alpha\Big\}+\begin{pmatrix}\mu\leftrightarrow\nu\\ k_1\leftrightarrow k_2\end{pmatrix}=8ie^2\,\varepsilon^{\mu\nu\alpha\beta}\,k_{1\beta}\,k_{2\alpha}\cdot\underbrace{\int\frac{d^4\ell}{(2\pi)^4}\cdot\frac{m^2}{(\ell^2-m^2)\left[(\ell+k_1)^2-m^2\right]\left[(\ell-k_2)^2-m^2\right]}}_{I_m}+\begin{pmatrix}k_1\leftrightarrow k_2\\ \mu\leftrightarrow\nu\end{pmatrix}$$

Approximate the integral by: ($\ell, m\sim$ large)

$$I_m\simeq\int\frac{d^4\ell}{(2\pi)^4}\,\frac{m^2}{\left[\ell^2-m^2+i\epsilon\right]^3}=\Bigg|\begin{array}{l}\text{Wick rotation}\\ \ell_0=+i\,\ell_0^E\end{array}$$



$$\ell^2-m^2+i\varepsilon=\left(\ell_0-\sqrt{\vec{\ell}^{\,2}+m^2}+i\varepsilon\right)\cdot\left(\ell_0+\sqrt{\vec{\ell}^{\,2}+m^2}-i\varepsilon\right)$$