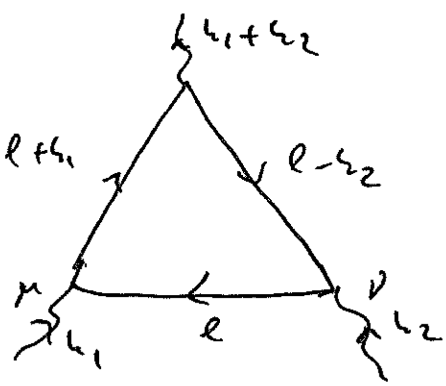
Last time | **Axial anomaly** (cont'd)

Want to check whether $\partial_\mu j^{5\mu} = 0$ holds at quantum level

$$T_{\mu\nu\rho} = i\int d^4x_1\, d^4x_2\, e^{ik_1\cdot x_1 + ik_2\cdot x_2} \langle 0|T\, j_\mu(x_1) j_\nu(x_2) j^5_\rho(0)|0\rangle$$



$$+ \begin{pmatrix} \mu \leftrightarrow \nu \\ k_1 \leftrightarrow k_2 \end{pmatrix}$$

$$(k_1+k_2)^\rho T_{\mu\nu\rho} = 8ie^2m^2\, \varepsilon^{\mu\nu\alpha\beta} k_{1\beta} k_{2\alpha} \int \frac{d^4\ell}{(2\pi)^4} \cdot \frac{1}{(\ell^2-m^2+i\varepsilon)\left[(\ell+k_1)^2-m^2+i\varepsilon\right]\left[(\ell-k_2)^2-m^2+i\varepsilon\right]} + \begin{pmatrix} \mu \leftrightarrow \nu \\ k_1 \leftrightarrow k_2 \end{pmatrix}$$

Pauli-Villars regularization mass

$\Rightarrow$ the $m=0$ term (first) in $[\ldots]$ vanishes like before.

For the term in $[\ldots]$ containing m^2 write:

$$Tr\left[\left(\not k_1+\not l-(\not l-\not k_2)\right)\gamma_5(\not l+\not k_1)\gamma_\mu\not l\gamma_\nu(\not l-\not k_2)\right]=$$

$$=-(l+k_1)^2\,Tr\left[\gamma_5\gamma_\mu\not l\gamma_\nu(\not l-\not k_2)\right]-(l-k_2)^2\,Tr\left[\gamma_5(\not l+\not k_1)\cdot\right.$$

$$\left.\cdot\gamma_\mu\not l\gamma_\nu\right]=-\left[(l+k_1)^2-m^2\right]\;Tr\left[\gamma_5\gamma_\mu\not l\gamma_\nu(\not l-\not k_2)\right]$$

$$-\left[(l-k_2)^2-m^2\right]Tr\left[\gamma_5(\not l+\not k_1)\gamma_\mu\not l\gamma_\nu\right]-m^2\left(Tr\left[\gamma_5\gamma_\mu\not l\gamma_\nu\cdot\right.\right.$$

$$\left.\left.\cdot(\not l-\not k_2)\right]+Tr\left[\gamma_5(\not l+\not k_1)\gamma_\mu\not l\gamma_\nu\right]\right)$$

First two terms also cancel after shifts.

We get:

$$(k_1+k_2)^\rho\,T_{\mu\nu\rho}=-e^2\int\frac{d^4l}{(2\pi)^4}\,m^2\,\frac{1}{\left[l^2-m^2\right]\left[(l+k_1)^2-m^2\right]\left[(l-k_2)^2-m^2\right]}$$

$$\cdot\Big\{Tr\left[\gamma_5\gamma_\mu\not l\gamma_\nu(\not l-\not k_2)\right]+Tr\left[\gamma_5(\not l+\not k_1)\gamma_\mu\not l\gamma_\nu\right]-$$

$$-Tr\left[(\not k_1+\not k_2)\gamma_5\gamma_\mu\gamma_\nu(\not l-\not k_2)\right]-Tr\left[(\not k_1+\not k_2)\gamma_5\cdot\right.$$

$$\left.\cdot(\not l+\not k_1)\gamma_\mu\gamma_\nu\right]-Tr\left[(\not k_1+\not k_2)\gamma_5\gamma_\mu\not l\gamma_\nu\right]\Big\}+\binom{k_1\leftrightarrow k_2}{\mu\leftrightarrow\nu}\quad m^2 \text{ terms in } Tr.$$

$$=\left(\text{as } Tr\left[\gamma^5\gamma^\mu\gamma^\nu\gamma^\alpha\gamma^\beta\right]=-4i\,\varepsilon^{\mu\nu\alpha\beta}\right)=$$

$$= 4ie^2 \varepsilon^{\mu\nu\alpha\beta} \int \frac{d^4 l}{(2\pi)^4} \frac{m^2}{(l^2-m^2)[(l+k_1)^2-m^2][(l-k_2)^2-m^2]}$$

$$\cdot \Big\{ -l_\alpha (\cancel{l}-k_2)_\beta + l_\alpha (\cancel{l}+k_1)_\beta - (l-k_2)_\alpha (k_1+k_2)_\beta + (k_1+k_2)_\alpha (l+k_1)_\beta$$

$$-(k_1+k_2)_\alpha l_\beta \Big\} + \begin{pmatrix} k_1 \leftrightarrow k_2 \\ \mu \leftrightarrow \nu \end{pmatrix} = 4ie^2 \varepsilon^{\mu\nu\alpha\beta} \int \frac{d^4 l}{(2\pi)^4} \cdot$$

$$\frac{m^2}{[l^2-m^2][(l+k_1)^2-m^2][(l-k_2)^2-m^2]} \Big\{ \cancel{l_\alpha k_{2\beta}} + \cancel{l_\alpha k_{1\beta}} -$$

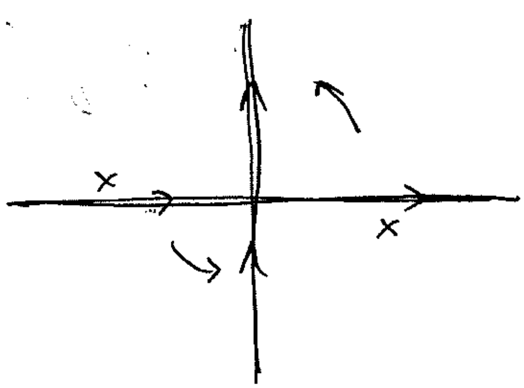
$$-\cancel{l_\alpha (k_1+k_2)_\beta} + \cancel{(k_1+k_2)_\alpha l_\beta} - \cancel{(k_1+k_2)_\alpha l_\beta} + k_{2\alpha} (k_1 + \underbrace{\cancel{k_2}}_{0})_\beta$$

$$+ k_{1\beta} (\underbrace{\cancel{k_1}}_{0} + k_2)_\alpha \Big\} + \begin{pmatrix} \mu \leftrightarrow \nu \\ k_1 \leftrightarrow k_2 \end{pmatrix} = 8ie^2 \varepsilon^{\mu\nu\alpha\beta} k_{1\beta} k_{2\alpha} \cdot \int \frac{d^4 l}{(2\pi)^4} \cdot$$

$$\cdot \underbrace{\frac{m^2}{(l^2-m^2 + i\varepsilon)[(l+k_1)^2-m^2+i\varepsilon][(l-k_2)^2-m^2+i\varepsilon]}}_{I_m} + \begin{pmatrix} k_1 \leftrightarrow k_2 \\ \mu \leftrightarrow \nu \end{pmatrix}$$

Approximate the integral by: $(l, m \sim \text{large})$

$$I_m \simeq \int \frac{d^4 l}{(2\pi)^4} \frac{m^2}{[l^2-m^2+i\varepsilon]^3} = \Bigg| \begin{array}{l} \text{Wick rotation} \\ l_0 = +i\, l_0^E \end{array}$$



$$l^2 - m^2 + i\varepsilon = \left(l_0 - \sqrt{\vec{l}^2+m^2} + i\varepsilon\right) \cdot \left(l_0 + \sqrt{\vec{l}^2+m^2} - i\varepsilon\right)$$

$$\Rightarrow I_m = -i\int \frac{d^4 l_E}{(2\pi)^4}\,\frac{m^2}{[l_E^2+m^2]^3} = -i\int_0^\infty \frac{l_E^3\,dl_E}{(2\pi)^4}\cdot\underbrace{\int d\Omega_4}_{2\pi^2}\cdot$$

$$\cdot\frac{m^2}{[l_E^2+m^2]^3} = -i\,\frac{1}{8\pi^2}\,m^2\int_0^\infty \frac{dl\cdot l^3}{[l^2+m^2]^3} = -i\,\frac{1}{16\pi^2}\,m^2\cdot$$

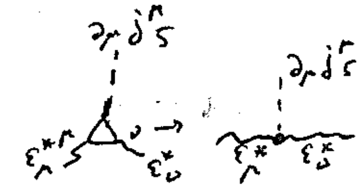
$$\cdot\int_0^\infty \frac{dl^2\cdot[l^2+m^2-m^2]}{[l^2+m^2]^3} = -i\,\frac{1}{(4\pi)^2}\,m^2\cdot\left[\frac{1}{m^2}-m^2\frac{1}{2m^4}\right] =$$

$$= -i\,\frac{1}{2}\,\frac{1}{(4\pi)^2}\,.$$ We get

$$\begin{pmatrix}\mu\leftrightarrow\nu\\ k_1\leftrightarrow k_2\end{pmatrix}$$

$$(k_1+k_2)^\rho\,T_{\mu\nu\rho} = 8\cancel{i}e^2\,\varepsilon^{\mu\nu\alpha\beta}\,k_{1\beta}\,k_{2\alpha}\,\cancel{(-i)}\,\frac{1}{2}\,\frac{1}{(4\pi)^2}\,2 \Rightarrow$$

$$(k_1+k_2)^\rho\,T_{\mu\nu\rho} = -2\frac{\alpha_{EM}}{\pi}\,\varepsilon^{\mu\nu\alpha\beta}\,k_{1\alpha}\,k_{2\beta}$$



$\Rightarrow$ in operator language this means:

$$\boxed{\partial_\mu j_5^\mu = \frac{-\alpha}{4\pi}\,\varepsilon^{\mu\nu\alpha\beta}\,F_{\mu\nu}\,F_{\alpha\beta}}$$ Adler-Bell-Jackiw anomaly '69

$\Rightarrow$ classically conserved current is <u>not</u> conserved quantum mechanically!

$\Rightarrow$ in QED this ABJ anomaly relation is exact $\sim$ no higher-order corrections.

In QCD have $j_{5\mu} = \sum_f \bar{q}_f \gamma_\mu \gamma_5 q_f$

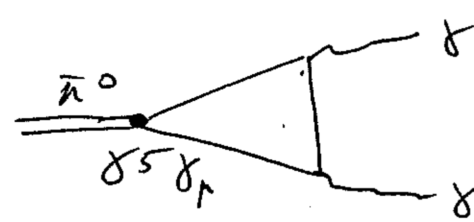
and $$\boxed{\partial_\mu j_5^\mu = -\frac{\alpha_s N_f}{8\pi} \varepsilon^{\mu\nu\alpha\beta} F^a_{\mu\nu} F^a_{\rho\sigma}}.$$

$\Rightarrow$ $U(1)_A$ in QCD is broken, but has no Goldstone boson associated with this breaking $\Rightarrow$ symmetry was never there in the full quantum theory

(Otherwise, if treating $U(1)_A$ as a symmetry, would expect parity-doubling of baryon states. If $U(1)_A$ is broken $\rightarrow$ expect Goldstone modes. This way we see that the symmetry is never a good symmetry.)

$\Rightarrow$ to get $\neq 0$ $\varepsilon^{\mu\nu\alpha\beta} F^a_{\mu\nu} F^a_{\rho\sigma}$ need instantons ...

$\Rightarrow$ axial anomaly is responsible for pion decay : $\pi^0 \rightarrow \gamma\gamma$

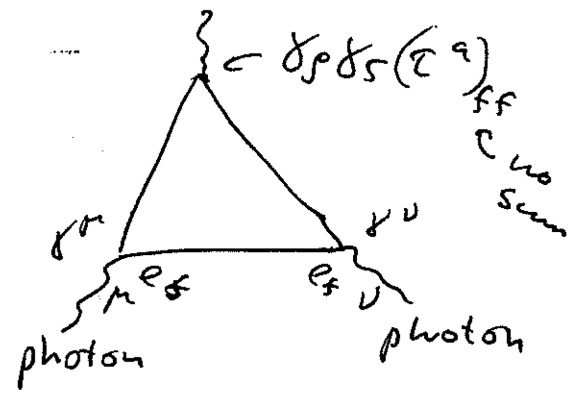


$\pi^0 \to \gamma\gamma$

Consider axial isospin current $j^a_{5\mu} = \bar{q}\gamma_\mu\gamma_5\tau^a q$

wher τ^a = Pauli matrices, $a = 1,2,3$ (flavor index for $SU(2)$ flavor). Here $q = \begin{pmatrix} u \\ d \end{pmatrix}$.

It has an anomaly due to quarks coupling to photons:



$$\partial_\mu j_5^{a\mu} = -\frac{\alpha_{EM}}{4\pi}\varepsilon^{\alpha\beta\mu\nu}F_{\alpha\beta}F_{\mu\nu} \cdot \sum_f (\tau^a)_{ff}\cdot e_f^2$$

as $\tau^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $\tau^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$, $\tau^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

$\Rightarrow$ only τ^3 gives $\neq 0$ anomaly

$$\sum_f (\tau^3)_{ff}\, e_f^2 = \left(\frac{2}{3}\right)^2 - \left(\frac{1}{3}\right)^2 = \frac{1}{3}$$

$$\Rightarrow \partial_\mu j_5^{3\mu} = -\frac{\alpha_{EM}}{12\pi}\varepsilon^{\mu\nu\alpha\beta}F_{\mu\nu}F_{\alpha\beta}$$

$j_5^{3\mu} = \bar{u}\gamma_\mu\gamma_5 u - \bar{d}\gamma_\mu\gamma_5 d$ annihilates π^0:

$\langle 0 | j_5^{3\mu}(0) | \pi^0(p)\rangle = i f_\pi p^\mu$ (due to spont. chiral symm. breaking)

axial charge doesn't annihil vac

with $f_\pi \simeq 93$ MeV (pion decay constant)

$\Rightarrow$ in general $\langle 0 | j_5^{3\mu}(x) | \pi^0(p) \rangle = i p^\mu f_\pi e^{-ip\cdot x}$

$\Rightarrow \langle 0 | \partial_\mu j_5^{3\mu}(x) | \pi^0(p) \rangle = \underbrace{p_\mu p^\mu}_{m_\pi^2} f_\pi e^{-ip\cdot x}$

$\Rightarrow \langle 0 | \partial_\mu j_5^{3\mu}(0) | \pi^0(p) \rangle = m_\pi^2 f_\pi$

$\Rightarrow$ pion couples to $\partial_\mu j_5^{3\mu} \sim \varepsilon^{\mu\nu\alpha\beta} F_{\mu\nu} F_{\alpha\beta}$

$\Rightarrow \sim A_\rho A_\sigma \Rightarrow$ pion couples to two photons

$\Rightarrow$ can have $\pi^0 \to \gamma\gamma$ decay due to the axial anomaly.

Axial anomaly in the Standard Model.

$\Rightarrow$ a theory with axial anomaly would violate Ward identities $\left((k_1+k_2)^\rho T_{\mu\nu\rho} = 0\right)$, an is therefore not gauge invariant!

$\Rightarrow$ this would be a problem for theories with axial current coupling to gauge bosons $\left(\begin{smallmatrix}\text{e.g.}\\ \text{SM}\end{smallmatrix}\right)$

$\Rightarrow$ in particular an anomaly would spoil renormalizability of the theory

$\Rightarrow$ Standard model has vector bosons coupling with γ_5 to leptons and quarks. For SM to be consistent need those 3-boson couplings with γ_5 to cancel!

Let's go back to SM lagrangian:

$$\begin{aligned}\mathcal{L} = {} & \bar{R}_e\, i\gamma^\mu \left(\partial_\mu + i\frac{g'}{2} Y B_\mu\right) R_e + \bar{L}_e\, i\gamma^\mu \left(\partial_\mu + i\frac{g'}{2} Y B_\mu - ig\,\frac{\vec{\tau}}{2}\cdot\vec{W}_\mu\right) L_e + (\mu,\tau) + \bar{L}_u\, i\gamma^\mu\left(\partial_\mu + i\frac{g'}{2} Y B_\mu - ig\,\frac{\vec{\tau}}{2}\cdot\vec{W}_\mu\right) \\ & \cdot L_u + \bar{R}_u\, i\gamma^\mu\left(\partial_\mu + i\frac{g'}{2} Y B_\mu\right) R_u + \bar{R}_d\, i\gamma^\mu\left(\partial_\mu + i\frac{g'}{2} Y B_\mu\right) R_d \\ & + (2 \text{ more generations}) + \ldots\end{aligned}$$

(we keep quark/lepton - vector boson terms only)

Y is the weak hypercharge

$$\boxed{Q = I_3 + \frac{Y}{2}}$$

Gell-Mann – Nishijima relation always holds.

$\Rightarrow$ for L_e : $I_3 = \pm 1/2$; $Q = 0$ for neutrino

$\Rightarrow$ $0 = \frac{1}{2} + \frac{Y}{2} \Rightarrow Y_{L_e} = -1$

for R_e have $I_3 = 0$, $Q = -1 \Rightarrow -1 = \frac{Y}{2} \Rightarrow Y_{R_e} = -2$

for L_u: u-quark has $Q = +2/3 \Rightarrow \frac{2}{3} = \frac{1}{2} + \frac{Y}{2}$

$\Rightarrow Y_{L_u} = 1/3$

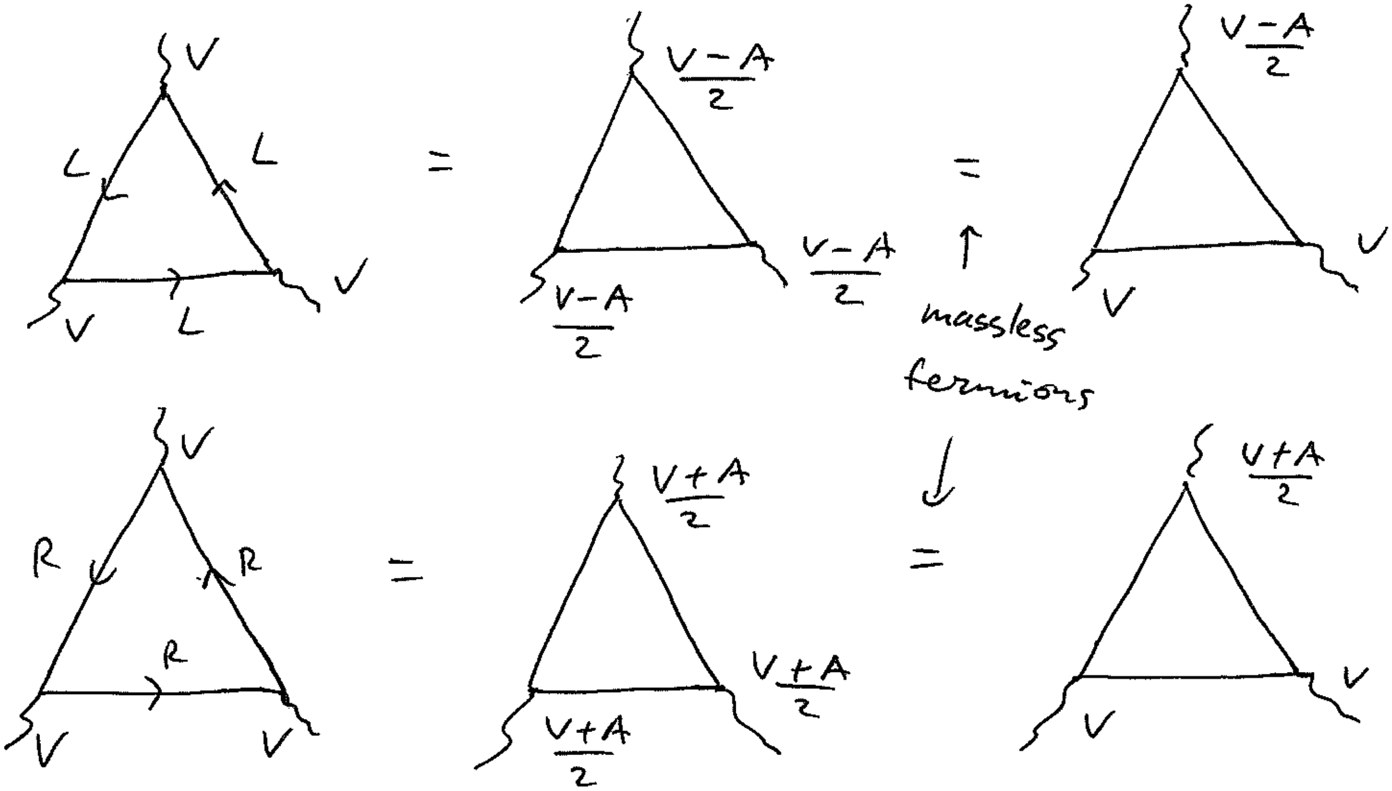
for R_u: $I_3 = 0 \Rightarrow \frac{2}{3} = \frac{Y}{2} \Rightarrow Y_{R_u} = 4/3$

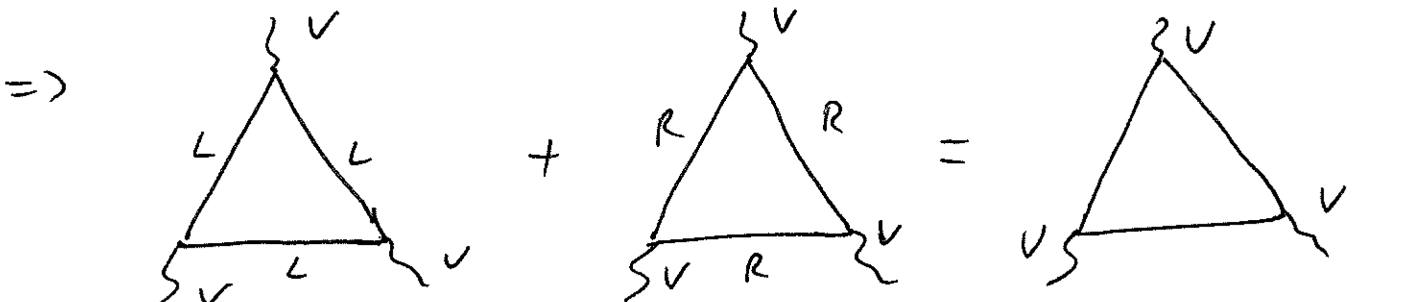
for R_d: $Q = -1/3 \Rightarrow -1/3 = \frac{Y}{2} \Rightarrow Y_{R_d} = -2/3$

other generations ~ same $\Rightarrow$ forget about them.

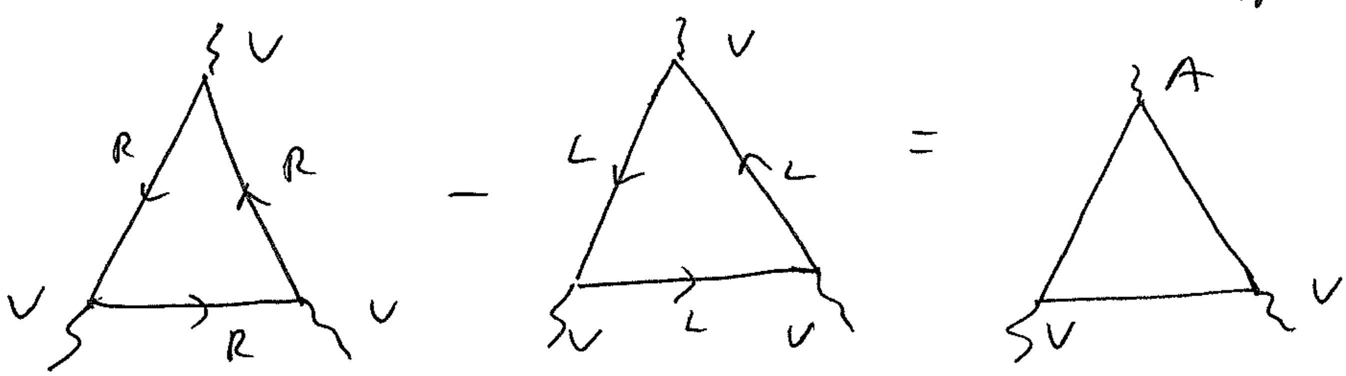
As $L_e = \begin{pmatrix} \nu_e \\ e \end{pmatrix}_L = \frac{1-\gamma_5}{2}\begin{pmatrix} \nu_e \\ e \end{pmatrix}$, $R_e = \frac{1+\gamma_5}{2} e = e_R$

$\Rightarrow$ all W_μ, B_μ couplings involve γ_5 $\Rightarrow$ need divergences to cancel.





as expected



the anomaly.

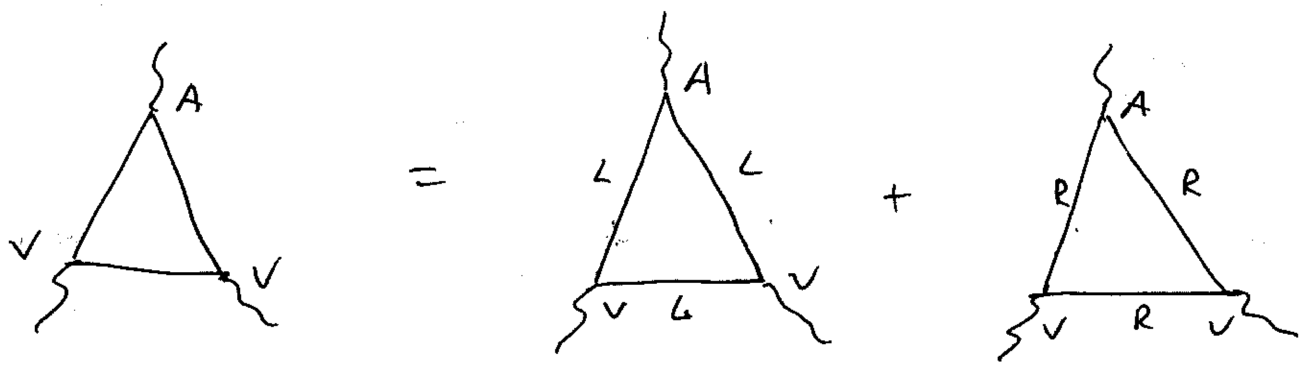
Anomaly is the difference between the right-handed and left-handed loops.

massless QED

$$\mathcal{L}_{QED} = \bar{\psi}\, i\gamma^\mu D_\mu \psi - \frac{1}{4} F_{\mu\nu} F^{\mu\nu} = \bar{\psi}_L\, i\gamma^\mu D_\mu \psi_L$$

$$+ \bar{\psi}_R\, i\gamma^\mu D_\mu \psi_R - \frac{1}{4} F_{\mu\nu} F^{\mu\nu}$$

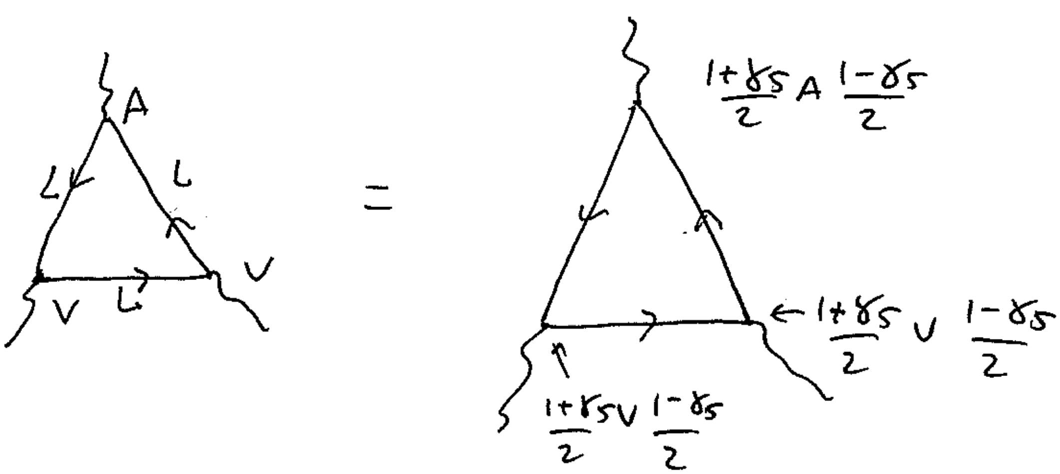
$\Rightarrow$ the anomaly consists of left-handed and right-handed (massless) electrons' contributions



$A = \gamma_\rho \gamma_5$

$V = \gamma_\mu$, or γ_ν

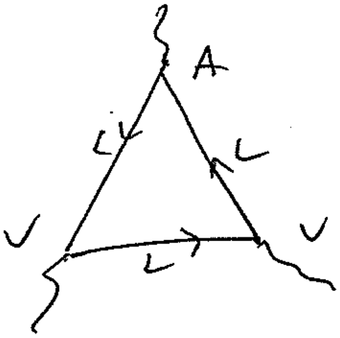
Propagator $\langle \psi_L \bar{\psi}_L \rangle = \langle \frac{1-\gamma_5}{2} \psi \bar{\psi} \frac{1+\gamma_5}{2} \rangle \Rightarrow$



$$\frac{1+\gamma_5}{2} \gamma_\mu \frac{1-\gamma_5}{2} = \gamma_\mu \frac{1-\gamma_5}{2} = \frac{V-A}{2}$$

$$\frac{1+\gamma_5}{2} \gamma_\rho \gamma_5 \frac{1-\gamma_5}{2} = \gamma_\rho \gamma_5 \frac{1-\gamma_5}{2} = \gamma_\rho \frac{\gamma_5 - 1}{2} = \frac{A-V}{2}$$

Hence



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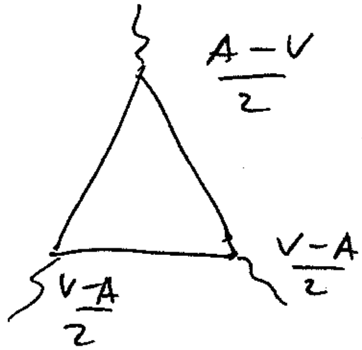


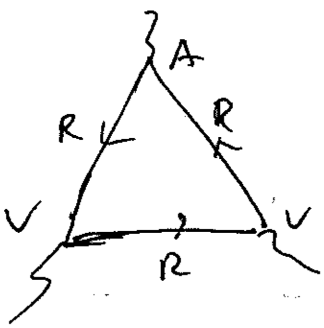
$= \frac{1}{2}$ Anomaly

Subtract, get

$= 0$

⇒ anomalies cancel!

Similarly



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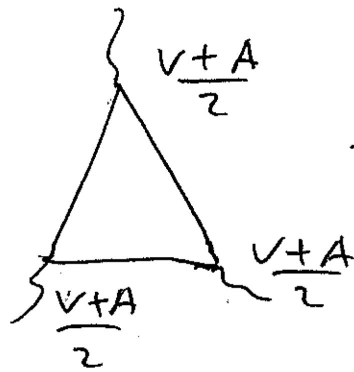
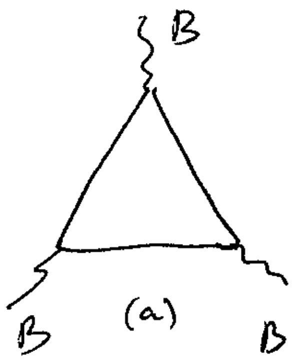


$= \frac{1}{2}$ Anomaly

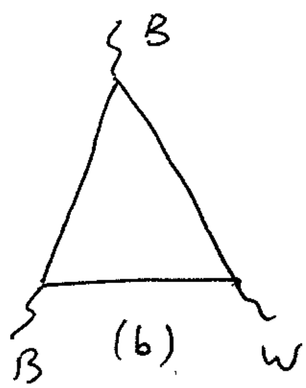
No anomaly in 3-boson coupling! (a QED)

⇒ in SM need to sum all graphs with left- and right- handed particles in the loop.

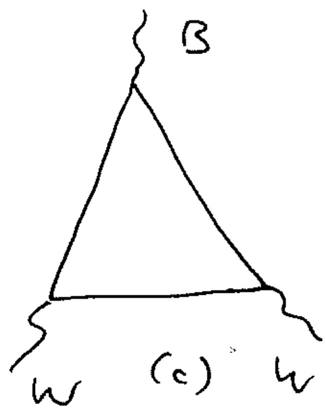
The diagrams are:



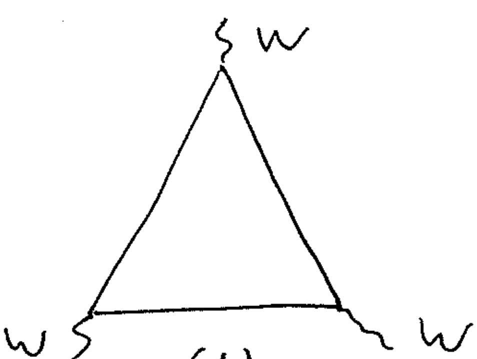
(a) ,



(b) ,

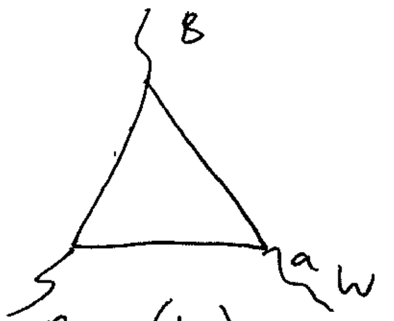


(c)



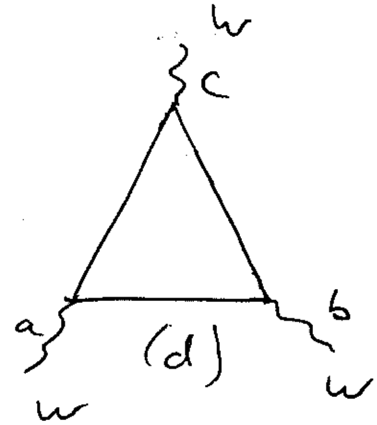
(d)

First let's do (b): $\text{tr}\,\tau^a = 0 \Rightarrow \boxed{(b) = 0.}$



(b)

Now, let's look at (d):

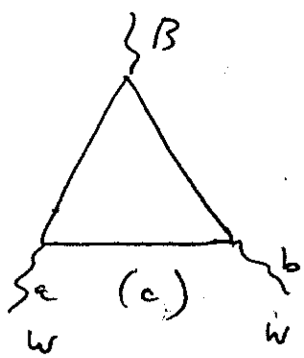


(d)

$$\text{tr}\left(\tau^c\tau^a\tau^b\right) + \text{tr}\left(\tau^c\tau^b\tau^a\right)$$

$$= \text{tr}\left[\tau^c \underbrace{\{\tau^a, \tau^b\}}_{2\delta^{ab}}\right] \sim \text{tr}\,\tau^c = 0 \Rightarrow \boxed{(d) = 0}$$

Next let's look at (c):



(c)

$\text{tr}\,\frac{\tau^a}{2}\frac{\tau^b}{2} = \frac{1}{2}\delta^{ab}$ is not zero

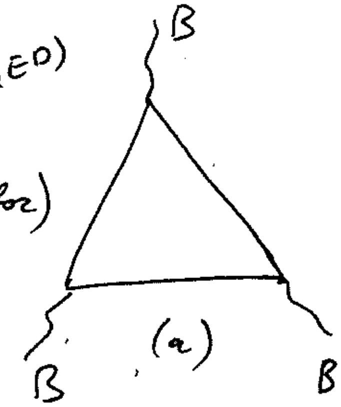
$$(c) \propto \sum_{\substack{i = \text{left-handed} \\ \text{doublets}}} Y_i$$ (as W couples to left-handed quarks & leptons only)

$$\Rightarrow (c) \propto Y_{Le} + Y_{Lu} \cdot \underset{\uparrow \text{ No. of colors}}{3} = -1 + \frac{1}{3} \cdot 3 = 0$$

$$\Rightarrow \boxed{(c) = 0}$$

Finally, let's look at (a):

contribute "–" to anomaly (see QED)



$$(a) \propto 2 \sum_{\substack{i = \text{left-handed} \\ \text{doublets}}} Y_i^3 \times (\text{color}) - \sum_{i = \text{right-handed}} Y_i^3 \otimes (\text{color})$$

doublets

$$= \underbrace{2(-1)^3}_{L_e} + \underbrace{2 \cdot \left(\frac{1}{3}\right)^3 \cdot \underset{\text{color}}{3}}_{L_u} - \underbrace{(-2)^3}_{R_e} - \underbrace{\left(\frac{4}{3}\right)^3 \cdot \underset{\text{color}}{3}}_{R_u} - \underbrace{\left(-\frac{2}{3}\right)^3 \cdot \underset{\text{color}}{3}}_{R_d}$$

$$= -2 + \frac{2}{9} + 8 - \frac{64}{9} + \frac{8}{9} = 6 - \frac{54}{9} = 0$$

$$\Rightarrow \boxed{(a) = 0}.$$

$\Rightarrow$ the same applies to the other two generations

$\Rightarrow$ anomalies cancel in 3-vector boson couplings in the SM! Thus Standard Model is a consistent (gauge-invariant) and renormalizable theory... as expected.