~~=> putting all this together write the lagrangian for Quantum Chromodynamics (QCD) – the theory of strong interactions:~~

$$\boxed{\cancel{\mathcal{L}_{QCD} = \bar{q}^{af}\left(i\gamma\cdot\partial - m_f\right)q^{af} - \frac{1}{4}F^{i}_{\mu\nu}F^{\mu\nu i} + g\,\bar{q}^{bf}\gamma^{\mu}A^{i}_{\mu}\left(T^{i}\right)_{ba}q^{af}}}$$

Elements of Group Theory

Def. A <u>Group</u> G is a set of elements with a multiplication law having the following properties:

(i) Closure: if $f, g \in G \Rightarrow h = f\cdot g \in G$

(ii) Associativity: $f, g, h \in G \Rightarrow f\cdot(g\cdot h) = (f\cdot g)\cdot h$

(iii) Identity: $\exists\, e \in G \;\; \forall f \in G : ef = fe = f$

(iv) Inverse element: $\forall f \in G \;\; \exists f^{-1} \in G : f f^{-1} = f^{-1} f = e$.

<u>Example</u>: $\{1, e^{i\frac{\pi}{2}}, e^{i\pi}, e^{i\frac{3}{2}\pi}\}$ form a group (why?). $\mathbb{Z}_4$ " $\{1, i, -1, -i\}$.

Integers: $\{\ldots, -2, -1, 0, 1, 2, \ldots\}$ form a group.

What is e there? **Def.** $H \subset G \Rightarrow H$ is a <u>subgroup</u>.

What is "multiplication" law?

Def. A group is called Abelian if for any $f, g \in G$: $f \cdot g = g \cdot f$

otherwise it is called non-Abelian ($f \cdot g \neq g \cdot f$)

Example (important!) | nxn unitary matrices form a group: $U U^{+} = U^{+} U = \mathbb{1}$ (unitary matrices).

Def. Such group is denoted $U(n)$, $\left(e = \mathbb{1} = \begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{pmatrix} \right)$
and is called the unitary group.

Sub-example | $U(1)$: 1×1 matrices $\Rightarrow$ $e^{i\varphi}$, $\varphi \sim \#$

$\varphi \in \mathbb{R}$ ~ form a group, $e = 1$.

Def. $n \times n$ unitary matrices with unit determinant ($U U^{+} = U^{+} U = \mathbb{1}$, $\det U = +1$) form a group too!

It is called special unitary group and is denoted $SU(n)$. (Orthogonal matrices $U^{T} U = U U^{T} = \mathbb{1}$ with $\det U = +1$ form $SO(n)$, O = orthogonal)

Def. A representation of group G is a mapping D of group elements : $f \in G$: $f \to D(f)$, where $D(f)$ is a space of linear operators (e.g. matrices) such that:

(i) $D(e) = \mathbb{1}$

(ii) $D(g_1) D(g_2) = D(g_1 g_2)$ for $g_1, g_2 \in G$.

Take a group $\mathbb{Z}_4$: it has $\{e, g_1, g_2, g_3\}$.

Our example $\{1, e^{i\frac{\pi}{2}}, e^{i\pi}, e^{i\frac{3\pi}{2}}\} = \{D(e), D(g_1), D(g_2), D(g_3)\}$

is one of the many possible representations of $\mathbb{Z}_4$.

Def. <u>Dimension</u> of representation is the dimension of the space of D-matrices.

Def. Representation is called <u>reducible</u> if $\exists M$ (matrix) such that

$$M\, D(g)\, M^{-1} = \begin{bmatrix} D_1(g) & 0 & \\ 0 & D_2(g) & \\ & & \ddots \end{bmatrix} \quad \text{for } \forall g \in G.$$

$\Rightarrow D = D_1 \oplus D_2 \oplus \ldots$

a representation is called <u>irreducible</u> if no such matrix M exists.

Def. For two groups $G = \{g_1, g_2, \ldots\}$, $H = \{h_1, h_2, \ldots\}$ define <u>direct-product</u> group $G \times H = \{g_i h_j\}$

such that $g_k h_e \cdot g_m h_n = g_k g_m \cdot h_e h_n$.

<u>Lie Groups</u>

Imagine a group G with elements smoothly dependent on a continuous set of parameters α_i, $i = 1, \ldots, N$: $g(\alpha_i) \in G$.

$\Rightarrow$ assume that $g(\alpha_i = 0) = e$ (the identity element)

$\Rightarrow$ for a representation of the group:

$$D(\alpha_i = 0) = \mathbb{1}.$$

Taylor expand $D(\alpha_i)$ near 0:

$$D(\delta\alpha_i) = \mathbb{1} + i\,\delta\alpha_i\,X_i + \dots = \mathbb{1} + i\,\delta\vec{\alpha}\cdot\vec{X}$$

(summation over repeating indices assumed)

Def. X_i are called <u>generators</u> of the group.

definition too $\qquad \delta\vec{\alpha} = \frac{\vec{\alpha}}{k}$, k ~ integer

$$D(\alpha_i) = D(\delta\alpha_i)\,D(\delta\alpha_i)\dots D(\delta\alpha_i) = \lim_{k\to\infty}\left(\mathbb{1} + i\,\delta\vec{\alpha}\cdot\vec{X}\right)^k$$

$$= \lim_{k\to\infty}\left(\mathbb{1} + i\,\frac{\vec{\alpha}}{k}\cdot\vec{X}\right)^k = e^{i\vec{\alpha}\cdot\vec{X}}.$$

Def. A group with elements depending smoothly on continuous set of parameters α_i, $i = 1, \dots, N$, with generators X_i is called a <u>Lie group</u>.

$D(\vec{\alpha}) = e^{i\vec{\alpha}\cdot\vec{X}}$ $\Rightarrow$ as D can be a matrix

$\Rightarrow$ $\vec{X}$ can be a matrix; therefore in general $\underbrace{[X_i, X_j]}_{= X_i X_j - X_j X_i}$ does not have to be 0.

$\Rightarrow$ however $D(\vec{\alpha})\,D(\vec{\beta}) = e^{i\vec{\alpha}\cdot X}\, e^{i\vec{\beta}\cdot X}$ is

also a group element $\Rightarrow e^{i\vec{\alpha}\cdot\vec{X}}\, e^{i\vec{\beta}\cdot\vec{X}} = e^{i\vec{\delta}\cdot\vec{X}}$

$\Rightarrow$ can show that for this to work we need

$$\boxed{[X_a, X_b] = i f_{abc} X_c}$$

Lie algebra of generators

f_{abc} ~ structure constants of the group

Def. Commutator:

$f_{abc} = - f_{bac}$;

$$[A,B] = A\cdot B - B\cdot A.$$

f_{abc} are <u>real</u> for unitary representations D

(for hermitean X_a): $D^{\dagger}D = DD^{\dagger} = \mathbb{1}$

Example take the group $SU(2)$: unitary 2×2 matrices with $\det = +1$ $(uu^{+} = u^{+}u = \mathbb{1}, \det u = +1)$.
(defining representation)
Using Pauli matrices we can define a representation of $SU(2)$:

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$\Rightarrow D(\vec{\alpha}) = e^{i\frac{\vec{\alpha}\cdot\vec{\sigma}}{2}}$, $\vec{\alpha} = (\alpha_1, \alpha_2, \alpha_3)$ a 3-vector.

rotations around $\frac{\vec{\alpha}}{|\vec{\alpha}|}$ axis by angle $|\vec{\alpha}|$.

as $\sigma_i^+ = \sigma_i$ (hermitean) $\Rightarrow$ any 2×2 unitary matrix with det $= +1$ can be represented as $e^{i\frac{\vec{\alpha}\cdot\vec{\sigma}}{2}} = U$.

Check: $UU^+ = e^{i\frac{\vec{\alpha}\cdot\vec{\sigma}}{2}}\, e^{-i\frac{\vec{\alpha}\cdot\vec{\sigma}}{2}} = \mathbb{1}$.

$$\det U = \det e^{i\frac{\vec{\alpha}\cdot\vec{\sigma}}{2}} = \left[\text{as } \det e^{A} = e^{\operatorname{tr} A}\right] = 1$$

as $\operatorname{tr}\sigma_i = 0$.

comp's cond's

$8 - 4 = 4 - 1 = 3$

(linearly independent)

$\Rightarrow$ there are $2^2 - 1 = 3$ different $n\times n$ traceless hermitean matrices $\Rightarrow \{\sigma_i\}$ use up all possibilities

Generators: $J_i = \frac{\sigma_i}{2} \Rightarrow D(\vec{\alpha}) = e^{i\vec{\alpha}\cdot\vec{J}}$.

$\Rightarrow SU(2)$ is a Lie group

We know that $[\sigma_i, \sigma_j] = 2i\,\varepsilon_{ijk}\,\sigma_k \Rightarrow$

$$\Rightarrow [J_i, J_j] = i\,\varepsilon_{ijk}\,J_k$$

$\Rightarrow$ generators of $SU(2)$ form a Lie algebra with structure constants ε_{ijk}

ε_{ijk} : totally anti-symmetric Levi-Civita symbol, $\varepsilon_{123} = 1$, $\varepsilon_{ijk} = -\varepsilon_{jik} = \varepsilon_{jki}\ \ldots$

$\varepsilon_{112} = 0\ \ldots$

<u>Another example:</u> $SU(3)$: 3×3 unitary matrices with $\det = +1$

Define Gell-Mann matrices:

$$\lambda_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \lambda_2 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \lambda_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \text{(cf. Pauli matrices)}$$

$$\lambda_4 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad \lambda_5 = \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix}, \quad \lambda_6 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix},$$

$$\lambda_7 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}, \quad \lambda_8 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}.$$

Normalization convention $\mathrm{tr}[\lambda_i \lambda_j] = 2\,\delta_{ij}$.

There are $3^2 - 1 = 8$ traceless hermitean 3×3 matrices

$\Rightarrow$ these should work.

Generators of $SU(3)$: $T^a = \dfrac{\lambda^a}{2}$ $\Rightarrow$

$\Rightarrow$ $[T^a, T^b] = i f^{abc} T^c$ with structure constants f^{abc}, which are anti-symmetric under the interchange of any two indices.

$\Rightarrow$ $SU(3)$ is a Lie group with the generator algebra given above.

a	b	c	f^{abc}
1	2	3	1
1	4	7	1/2
1	5	6	-1/2
2	4	6	1/2
2	5	7	1/2
3	4	5	1/2
3	6	7	-1/2
4	5	8	$\sqrt{3}/2$
6	7	8	$\sqrt{3}/2$

$f_{112} = 0$ ...

all other f^{abc}'s can be obtained from this table.

Casimir operator commutes with all generators:

$\vec{T}^2 = T_1^2 + T_2^2 + \dots + T_n^2 = \frac{N^2-1}{2N}$

$\Rightarrow$ for $SU(2)$ it is $3/4$

for $SU(3)$ it is $4/3$.

$$D(\vec{A}) = e^{i\vec{A}\cdot\vec{T}}, \text{ with } \vec{A} = (A_1, A_2, \dots, A_8)$$

~ an 8-component vector.

Jacobi Identity and the Adjoint Representation

~ go back to some general Lie group with the generators X_a obeying some Lie algebra

$$[X_a, X_b] = i f_{abc} X_c .$$

One can then easily prove Jacobi identity:

$$[X_a, [X_b, X_c]] + [X_b, [X_c, X_a]] + [X_c, [X_a, X_b]] = 0.$$

(prove this by using definitions of commutators)

$\Rightarrow$ plug in the commutator of Lie algebra to write

$$f_{bdc} f_{ade} + f_{abd} f_{cde} + f_{cad} f_{bde} = 0$$

this relations are obeyed by structure constants of any Lie group, e.g. $SU(n)$.

Define The generators in the adjoint representation by $(t^a)_{bc} = -i f_{abc}$ $\Rightarrow$ the above relation gives

$$[t^a, t^b] = i f_{abc}\, t^c$$

$\Rightarrow$ they obey the Lie algebra too!

Def. $D(\vec{A}) = e^{i A^a t^a}$ gives the <u>adjoint</u> representation of Lie group. (irreducible representation)

Lorentz Group

Work in Minkowski space, $\eta_{\mu\nu} = \eta^{\mu\nu} = \begin{pmatrix} 1 & & & 0 \\ & -1 & & \\ & & -1 & \\ 0 & & & -1 \end{pmatrix}$

$\eta_{\mu\nu}\eta^{\nu\rho} = \delta_\mu{}^\rho$; $\quad x_\mu = \eta_{\mu\nu} x^\nu = (t, -\vec{x})$, $x^\mu = (t, \vec{x})$.

Def. Set of linear (real) transformations

$$x^\mu \to x'^\mu = \Lambda^\mu{}_\nu x^\nu$$

forms the Lorentz group if

$$\eta_{\mu\nu} x'^\mu x'^\nu = \eta_{\mu\nu} x^\mu x^\nu$$

(proper time is preserved).

$\eta_{\mu\nu} = \begin{pmatrix} 1 & & & 0 \\ & -1 & & \\ & & -1 & \\ 0 & & & -1 \end{pmatrix}$ metric tensor

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Example $\quad \Lambda^\mu{}_\nu = \begin{pmatrix} \gamma & -\beta\gamma & 0 & 0 \\ -\beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$ for boosts along x^1-axis.

$$\eta_{\mu\nu} x'^\mu x'^\nu = \eta_{\mu\nu} x^\mu x^\nu$$

$$\eta_{\mu\nu} \Lambda^\mu{}_\alpha \Lambda^\nu{}_\beta x^\alpha x^\beta = \eta_{\alpha\beta} x^\alpha x^\beta$$

$$\Rightarrow \quad \boxed{\eta_{\alpha\beta} = \eta_{\mu\nu} \Lambda^\mu{}_\alpha \Lambda^\nu{}_\beta}$$

or, equivalently, $\quad \eta = \Lambda^T \eta \Lambda$

as $\eta_{\mu\nu}\eta^{\nu\rho} = \delta_\mu{}^\rho \Rightarrow \eta \cdot \eta = \mathbb{1}$