Spontaneous (Chiral) Symmetry Breaking.

General Discussion: Spontaneous Symmetry Breaking.

**Def.** Spontaneous Symmetry Breaking (SSB): a symmetry which is manifest in Lagrangian (and Hamiltonian), but is not respected by the ground state of the system.

**Example:** Ising model in 2d: \( H = -\sum_{\text{nearest neighbors}} J \overleftrightarrow{s_i s_j} \)

\( J > 0 \Rightarrow \text{spins tend to align} \)

\( s_i = \pm 1 \approx \text{projection of spins on y-axis} \)

\( \Rightarrow \text{the system is up-down symmetric:} \)

\( H \text{ is invariant under } s_i \rightarrow -s_i \).

\( \Rightarrow \text{however, the system spontaneously chooses a ground state, which is either all spins up or all spins down.} \)

\[ \begin{align*}
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\uparrow \uparrow \uparrow \\
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\downarrow \downarrow \downarrow \\
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\downarrow \downarrow \downarrow \\
\end{align*} \]
In this ground state one has \( \langle s_i \rangle \neq 0 \) and a nonzero magnetization \( \Rightarrow S_z \neq -S_z \) invariance is lost.

(Note that \( S_z \rightarrow -S_z \) is still a symmetry of \( H \) !)

Landau - Ginzburg theory of ferromagnetism:

\[
H = \int d^3x \left[ \left( \nabla \cdot M \right)^2 + \mu^2 \left( T - T_c \right) \vec{M}^2 + \lambda \left( \vec{M}^2 \right)^2 \right] + V(\vec{M}) \quad \text{the potential.}
\]

\( \vec{M} = (M_1, M_2, M_3) \) is magnetization of the medium.

\( \lambda > 0 \) constant, \( \mu^2 > 0 \) constant.

\( T \) - temperature, \( T_c \) - critical (Curie) temperature.

Let's plot the potential \( V(\vec{M}) \): (assume 2d system)

\( \Rightarrow \) the Hamiltonian is symmetric under spatial rotations: \( M_i \rightarrow M'_i = R \delta M_i \).
\( x_i' = R_{ij} x_j \Rightarrow \| x' \|^2 = \| x \|^2 \Rightarrow R_{ij} x_j x_k = x_i x_c \)

\( \Rightarrow R_{ij} R_{ik} = \delta_{jk} \Rightarrow R R^T = R^T R = I \Rightarrow \text{for get reflections} \Rightarrow \text{ require det } R = +1 \Rightarrow SO(3) \)

a group of special (det = +1) orthogonal (\( R R^T = R^T R = I \)) 3 x 3 matrices.

\( \Rightarrow \text{ for } T < T_c \text{ the ground state is at the minima} \)

\( \Rightarrow \mu^2 (T - T_c) 2 |\vec{M}| + 4 \lambda |\vec{M}|^3 = 0 \)

\[ |\vec{M}_{\text{vac}}| = \sqrt{\frac{\mu^2 (T_c - T)}{2 \lambda}} \]

\( \Rightarrow \text{ however, direction of } \vec{M} \text{ is chosen spontaneously!} \)

Say, \( M_{\text{vac}} = \sqrt{\frac{\mu^2 (T_c - T)}{2 \lambda}} \hat{x} = |0\rangle \)

Define generators of \( SO(3) \): \( L_1 = (0 \ 0 \ i) \), \( L_2 = (0 \ 0 \ 0 \ 0 \ i) \), \( L_3 = (0 \ i \ 0 \ 0) \Rightarrow e^{-i \hat{L}_3} \hat{L} \) is a rotation by angle \( \theta \) around \( \hat{z} \)-direction.

\( \Rightarrow \hat{H} \text{ is invariant under } \vec{M} \rightarrow \vec{M}' = e^{-i \hat{L}_3} \vec{M} \)

\( \Rightarrow \text{ ground state is not rotationally symmetric:} \)

\( R |0\rangle \neq |0\rangle \Rightarrow i \text{ if } R = e^{i \hat{L} \cdot \hat{Q}}, \hat{Q} \text{ a conserved charges of symmetry} \Rightarrow \vec{Q} |0\rangle \neq 0 \) (equivalently)
Imagine a system with Hamiltonian $H$ and conserved symmetry charges $Q^i : [H, Q^i] = 0$. Act on vacuum: $H|0\rangle = 0$ (can choose vacuum to be 0-energy state)

$$H Q^i |0\rangle = [H, Q^i] |0\rangle + Q^i H |0\rangle = 0$$

$$\Rightarrow H Q^i |0\rangle = 0 \Rightarrow \text{either}$$

(i) $Q^i |0\rangle = 0 \Rightarrow$ no broken symmetries, vacuum is invariant under $Q^i : e^{i\vec{q} \cdot \vec{Q}} |0\rangle = |0\rangle$.

(ii) $Q^i |0\rangle \neq 0 \Rightarrow$ vacuum is degenerate, more than one state such that $H |\psi_0\rangle = 0$.

(e.g. rotating ground state in $\lambda - G$ model would give other possible ground states)

$\Rightarrow$ if the system spontaneously chooses one of these $|\psi_0\rangle$ states for its ground state $\Rightarrow$ spontaneous symmetry breaking.