Quantum Chromodynamics (QCD): theory of quarks and gluons. SU(3) gauge group

Quark fields: q_α^{if} ← color, $i = 1, 2, 3$; f ← flavor index, $f = u, d, s, c, b, t$; α ← spinor index, $\alpha = 1, 2, 3, 4$

A_μ^a ~ gluon fields; a ← color, $a = 1, \ldots, 8$; μ ← Lorentz index $\mu = 0, 1, 2, 3$

The Lagrangian is

$$\boxed{\mathcal{L}_{QCD} = \bar{q}^{if}\left[i\gamma \cdot D_{ij} - m_f \right] q^{jf} - \frac{1}{4} F_{\mu\nu}^a F^{a\mu\nu}}$$

$D_\mu = \partial_\mu - i g A_\mu^a t^a$, $\quad t^a = \frac{\lambda^a}{2}$ ~ Gell-Mann matrices

⇒ Sum over flavors and colors assumed.

⇒ Other local non-Abelian theories in nature:

electroweak interactions (SU(2) group).

Last time | Functional Integral Quantization

(cont'd)

Path Integral Quantum Mechanics (cont'd)

Non-relativistic 1-particle QM: $\hat{H} = \frac{\hat{p}^2}{2m} + V(\hat{q})$

$[\hat{q}, \hat{p}] = i\hbar$, $\psi(q,t) = {}_S\langle q(t) | \psi(t) \rangle_S$ (wave function)

$$\Rightarrow \psi(q,t) = \int_{-\infty}^{\infty} dq' \, \langle q(t) | e^{-\frac{i}{\hbar}\hat{H}(t-t')} | q'(t') \rangle \, \psi(q',t')$$

Def. Time-evolution (Feynman) kernel:

$$U(q,t;q',t') \equiv {}_S\langle q(t) | e^{-\frac{i}{\hbar}\hat{H}(t-t')} | q'(t') \rangle_S .$$

We showed that

$$U(q,t;q',t') = \lim_{N\to\infty} \int_{-\infty}^{\infty} \left[\prod_{i=1}^{N-1} \frac{dq_i \, dp_i}{2\pi\hbar} \right] \frac{dp_N}{2\pi\hbar} \, e^{\frac{i}{\hbar}\delta t \sum_{j=1}^{N} \left[p_j \frac{q_j - q_{j-1}}{\delta t} - H \right]}$$

& denoted this object by

$$U(q,t;q',t') = \int [\mathcal{D}q \, \mathcal{D}p] \, e^{\frac{i}{\hbar}\int_{t'}^{t} dt'' [p(t'')\dot{q}(t'') - H(p(t''),q(t''))]}$$

path integral.

Integrating out $\mathcal{D}p$ we got

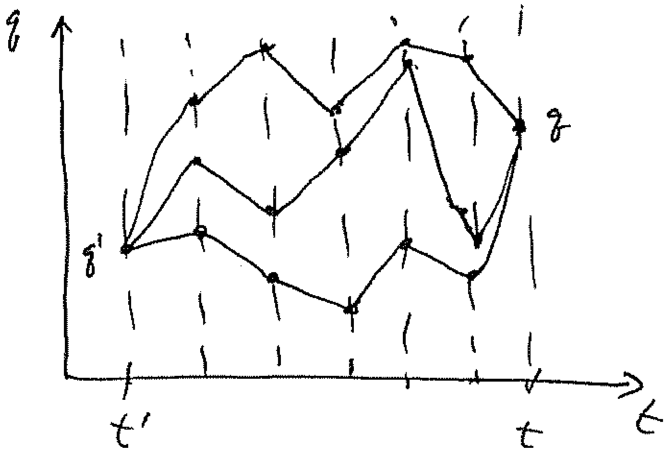
$$U(q,t;q',t') = \lim_{N\to\infty} \int_{-\infty}^{\infty} \left[\prod_{i=1}^{N-1} dq_i \right] \left[\frac{m}{2\pi\hbar i \delta t} \right]^{N/2} e^{\frac{i}{\hbar}\delta t \sum_{j=1}^{N} L(q_j, \dot{q}_j)}$$

which we denoted by

$$U(q,t;q',t') = \mathcal{N} \int [\mathcal{D}q]\, e^{\frac{i}{\hbar}\int_{t'}^{t} dt''\, L(q(t''),\, \dot{q}(t''))}$$

$$= \mathcal{N} \int [\mathcal{D}q]\, e^{\frac{i}{\hbar} S(q,t;q',t')}$$

integral over all paths:



Example | Free particle, $V(q) = 0$

$$U(q,t;q',t') = \lim_{N\to\infty} \left[\frac{m}{2\pi i \hbar \delta t}\right]^{N/2} \int_{-\infty}^{\infty} \left[\prod_{i=1}^{N-1} dq_i\right] e^{\frac{i}{\hbar}\delta t \frac{m}{2}\sum_{j=1}^{N} \frac{(q_j - q_{j-1})^2}{\delta t^2}}$$

$\Rightarrow$ integrated out q_1

Last time | Harmonic oscillator:

$$\boxed{U_{ho}(q_f, t_f; q_i, t_i) = \sqrt{\frac{m}{2\pi i \hbar T}} \sqrt{\frac{\omega T}{\sin \omega T}} \, e^{\frac{i}{\hbar} S_{cl}}.}$$

Time-ordered product: $\hat{q}_H(t) = e^{\frac{i}{\hbar}\hat{H}t} \, \hat{q}_S \, e^{-\frac{i}{\hbar}\hat{H}t}$

$$|q, t\rangle_H = e^{\frac{i}{\hbar}\hat{H}t} |q(t)\rangle_S$$

$$\boxed{U(q_f, t_f; q_i, t_i) = {}_H\langle q_f, t_f | q_i, t_i\rangle_H} = N \int [\mathcal{D}q] \, e^{\frac{i}{\hbar} S}.$$

$$\boxed{{}_H\langle q_f, t_f | T \hat{q}_H(t_2) \, \hat{q}_H(t_1) | q_i, t_i\rangle_H = N \int [\mathcal{D}q] \, q(t_2) \, q(t_1) \, e^{\frac{i}{\hbar} S}}$$

$$\boxed{{}_H\langle q_f, t_f | T \hat{q}_H(t_1) \dots \hat{q}(t_N) | q_i, t_i\rangle_H = N \int [\mathcal{D}q] \, q(t_1) \dots q(t_N) \cdot e^{\frac{i}{\hbar} S}}$$

Vacuum-to-vacuum tr. amplitude:

$$Z[j] \propto {}^{j}\langle 0, +\infty | 0, -\infty\rangle^{j}$$

$$L \to L + \hbar j q$$

$$\boxed{Z[j] = \int [\mathcal{D}q] \, e^{\frac{i}{\hbar} \int_{-\infty(1-i\delta)}^{\infty(1+i\delta)} dt \, [L + \hbar j(t) q(t)]}.}$$

Last time | Functional Quantization of the Scalar Field Theory (cont'd)

By analogy with QM, we introduced generating functional for Green functions:

$$Z[j] = \int \mathcal{D}\varphi \, e^{i\int d^4x \left[\mathcal{L} + j(x)\varphi(x)\right]}$$

where $\mathcal{L} = \frac{1}{2}\partial_\mu\varphi\,\partial^\mu\varphi - \frac{m^2}{2}\varphi^2 + \mathcal{L}_{int}$.

Any n-point Green function is then

$$\langle \Psi_0 | T\{\varphi_H(x_1)\ldots\varphi_H(x_n)\} | \Psi_0\rangle = \frac{\int \mathcal{D}\varphi\; \varphi(x_1)\ldots\varphi(x_n)\, e^{i\int d^4x\,\mathcal{L}}}{\int \mathcal{D}\varphi\, e^{i\int d^4x\,\mathcal{L}}}$$

or, equivalently,

$$\langle \Psi_0 | T\{\varphi_H(x_1)\ldots\varphi_H(x_n)\} | \Psi_0\rangle = (-i)^n \frac{1}{Z[j=0]} \left.\frac{\delta^n Z[j]}{\delta j(x_1)\ldots\delta j(x_n)}\right|_{j=0}$$

Free Scalar Theory

$$\mathcal{L}_0 = \frac{1}{2}\partial_\mu\varphi\,\partial^\mu\varphi - \frac{m^2}{2}\varphi^2 \Rightarrow Z_0[j] = \int \mathcal{D}\varphi \cdot e^{i\int d^4x\left[\frac{1}{2}\partial_\mu\varphi\partial^\mu\varphi - \frac{m^2 - i\varepsilon}{2}\varphi^2 + j\varphi\right]}.$$

We showed that

$$\boxed{\int (dx)\, e^{-\frac{1}{2}x^T A x} = \frac{1}{\sqrt{\det A}}}, \quad (dx) = \frac{d^n x}{(2\pi)^{n/2}},$$

$A \sim$ symmetric real ($\Rightarrow$ invertible)

$\Rightarrow$ by defining φ_0 using $\left(\Box + m^2 - i\varepsilon\right)\varphi_0 = j$ & the above formula we proved that

$$Z_0[j] = \frac{1}{\sqrt{\det(i\hat{D})}} \, e^{\frac{i}{2}\int d^4x \, j \cdot \varphi_0}$$

where $\hat{D} = \Box + m^2 - i\varepsilon$

$$\Rightarrow \boxed{Z_0[j] = \frac{1}{\sqrt{\det(i\hat{D})}} \, e^{-\frac{1}{2}\int d^4x\, d^4y\, j(x)\, D_F(x-y)\, j(y)}}$$

where $\boxed{D_F(x-y) = \int \frac{d^4p}{(2\pi)^4} \, e^{-ip\cdot(x-y)} \frac{i}{p^2 - m^2 + i\varepsilon}}$.

(Feynman propagator)

$\left(\Box + m^2\right) D_F(x-y) = -i\,\delta^4(x-y)$ $\Rightarrow$ D_F is a Green function of $(\Box + m^2)$.

$\Rightarrow$ any correlation function can be written as

$$\langle \varphi_0 | T\, \varphi(x_1) \ldots \varphi(x_{2n}) | \varphi_0 \rangle = \left(\left(-i \frac{\delta}{\delta j(x_1)}\right) \ldots \left(-i \frac{\delta}{\delta j(x_{2n})}\right) \frac{Z_0[j]}{Z_0[0]} \right)_{j=0}$$

$\Rightarrow$ just a product of propagators $\Rightarrow$ get Feynman diagrams this way.

Last time | Functional integral quantization (φ^4)

Free theory:

$$Z_0[j] = \int \mathcal{D}\varphi \; e^{i\int d^4x \left[\frac{1}{2}\partial_\mu\varphi\partial^\mu\varphi - \frac{m^2}{2}\varphi^2 + j\varphi\right]}$$

$$\Rightarrow \boxed{Z_0[j] = \frac{1}{\sqrt{\det(i\hat{D})}}\, e^{-\frac{1}{2}\int d^4x\, d^4y\, j(x) D_F(x-y) j(y)}}$$

where $\hat{D} = \Box + m^2 - i\varepsilon$, $D_F(x-y) = \int \frac{d^4p}{(2\pi)^4} e^{-ip\cdot(x-y)} \frac{i}{p^2-m^2+i\varepsilon}$

Feynman propagator

Interacting theory:

$$\boxed{\begin{aligned} Z[j] &= \int \mathcal{D}\varphi\, e^{i\int d^4x \left[\frac{1}{2}\partial_\mu\varphi\partial^\mu\varphi - \frac{m^2}{2}\varphi^2 - \frac{\lambda}{4!}\varphi^4 + j\varphi\right]} \\ &= e^{-i\frac{\lambda}{4!}\int d^4x \frac{\delta^4}{\delta j(x)^4}}\; Z_0[j] \end{aligned}}$$

$\Rightarrow$ expand in λ $\Rightarrow$ get Feynman diagrams

$$\langle \psi_0 | T\varphi(x_1)\varphi(x_2)|\psi_0\rangle = \frac{1}{Z[0]}(-i)^2 \frac{\delta^2}{\delta j(x_1)\delta j(x_2)} Z[j]\Big|_{j=0}$$

↑ this is how you find expectation values.

Faddeev - Popov Quantization

We want to quantize a gauge theory:

$$\mathcal{L} = -\frac{1}{4} F^a_{\mu\nu} F^{a\mu\nu}$$ (consider a general non-Abelian case).

The generating functional is

$$Z[0] = \int \mathcal{D} A_\mu \, e^{iS} = \int \mathcal{D} A_\mu \, e^{i \int d^4x \left(-\frac{1}{4}\right) F^a_{\mu\nu} F^{a\mu\nu}} =$$

$$= \int \mathcal{D} \bar{A}_\mu \, e^{iS} \cdot \int \mathcal{D} \Lambda$$

where $\bar{A}_\mu$ is the field in one particular gauge, Λ is the gauge transformation.

Problem: $\int \mathcal{D}\Lambda = \infty \Rightarrow Z = \infty \Rightarrow$ bad!

Even worse is the need to pick a gauge: consider abelian field A_μ: $\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} = -\frac{1}{4} (\partial_\mu A_\nu - \partial_\nu A_\mu)(\partial^\mu A^\nu - \partial^\nu A^\mu) = \frac{1}{2} A^\mu \underbrace{[g_{\mu\nu} \partial^2 - \partial_\mu \partial_\nu]}_{(D^{-1})_{\mu\nu}} A^\nu$

$\Rightarrow$ to find photon propagator need to solve

$$-i [g_{\mu\nu} \partial^2 - \partial_\mu \partial_\nu] D^{\nu\rho}(x) = \delta_\mu^\rho \, \delta^4(x)$$

$\Rightarrow$ act with $\partial^\mu \Rightarrow -i(\partial_\nu \partial^2 - \partial^2 \partial_\nu) D^{\nu\rho} = 0 = \partial^\rho \delta^4(x)$

$\Rightarrow$ this can not be true $\Rightarrow$ the operator has no inverse! $\Rightarrow$ no photon propagator?

However, if we choose a gauge, e.g. $\partial_\mu A_\mu = 0$

$$\Rightarrow \quad \mathcal{L} = \frac{1}{2} A^\mu \Box A^\nu \Rightarrow -i g_{\mu\nu} \Box D^{\nu\rho}(x) = \delta_\mu{}^\rho \delta^4(x).$$

$\Rightarrow$ easy to invert!

$\Rightarrow$ Need to fix the gauge!

Start with $\quad Z(0) = \int \mathcal{D} A_\mu \, e^{iS}$.

Insert into the integrand $\left(A_\mu^\Lambda = \Lambda A_\mu \Lambda^+ - \frac{i}{g}(\partial_\mu \Lambda)\Lambda^+\right)$

$$1 = \int \mathcal{D}\Lambda \; \delta(\Lambda - \mathbb{1}) = \int \mathcal{D}\Lambda \cdot \delta(G(A^\Lambda)) \det\left(\frac{\delta G(A^\Lambda)}{\delta \Lambda}\right)$$

eval. at $\Lambda = \mathbb{1}$ due to δ-fct.

where $G(A) = 0$ is the gauge condition we want to impose, e.g. $G(A) = \partial_\mu A^\mu$ for covariant gauge. Now

$$Z(0) = \int \mathcal{D} A_\mu \, e^{iS(A_\mu)} \int \mathcal{D}\Lambda \; \delta(G(A^\Lambda)) \det\left(\frac{\delta G(A^\Lambda)}{\delta \Lambda}\right)\Bigg|_{\Lambda = \mathbb{1}}$$

Change the order of integration & define a new field $A_\mu' = A_\mu^\Lambda$ to write (dropping the prime)

(as in QED $\mathcal{D} A_\mu = \mathcal{D} A_\mu^\Lambda$, $\underline{S(A_\mu) = S(A_\mu^\Lambda)}$) Λ = unitary