Last time | $q_f + q_{f'} \to q_f + q_{f'}$

$$\boxed{\frac{d\sigma^{f+f'\to f+f'}}{dt} = \frac{\pi \alpha_s^2}{s^2} \frac{C_F}{N_c} \frac{s^2+u^2}{t^2}}$$

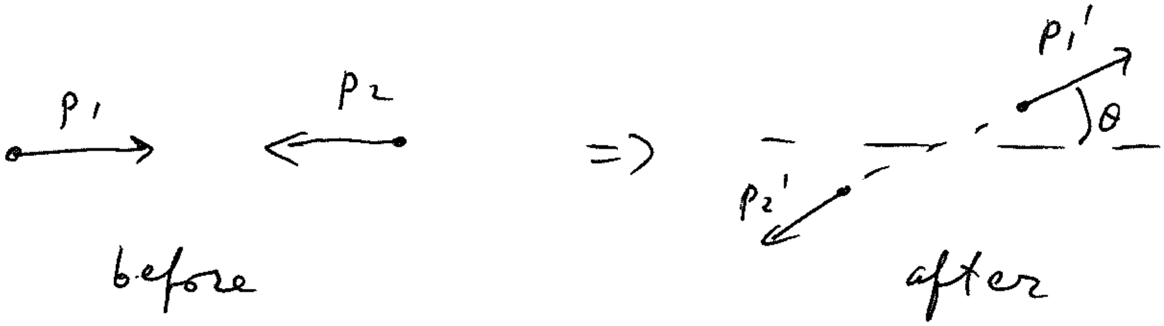
~ learned how to calculate color factors.

~ $t = \text{const}, \; s \to \infty$ (Regge limit)

$$\frac{d\sigma^{f+f'\to f+f'}}{dt} \simeq 2\pi \alpha_s^2 \frac{C_F}{N_c} \frac{1}{t^2}$$

~ does not decrease with the energy.

due to gluon exchange in t-channel



$$t = (p_1 - p_1')^2 \underset{\substack{\uparrow \\ \text{massless} \\ \text{quarks}}}{=} -2 p_1 \cdot p_1' = -2\left(E_1 E_1' - \vec{p}_1 \cdot \vec{p}_1'\right) =$$

$$= -2 E_1^2 (1 - \cos\theta) \leq 0$$

note, $\boxed{t \leq 0}$

$$dt = -2 E_1^2 \sin\theta \, d\theta = 2 E_1^2 \, d\cos\theta$$

$$\Rightarrow d\cos\theta = \frac{dt}{2E_1^2}$$

$$d\sigma = \frac{1}{8E_1^2} \frac{1}{16\pi} \cdot \frac{dt}{2E_1^2} \langle |M|^2 \rangle$$

$$s = (p_1 + p_2)^2 = 2 p_1 \cdot p_2 = 2\left(E_1 E_2 - \vec{p}_1 \cdot \vec{p}_2\right) = 4 E_1^2$$

$$\Rightarrow E_1^2 = s/4 \Rightarrow d\sigma = \frac{1}{2s} \frac{1}{16\pi} 2 \frac{dt}{s} \langle |M|^2 \rangle$$

$$\Rightarrow \frac{d\sigma}{dt} = \frac{1}{16\pi s^2} \langle |M|^2 \rangle = \frac{1}{16\pi s^2} \frac{g^4}{2t^2} \frac{N_c^2 - 1}{N_c^2} (s^2 + u^2)$$

$$\underset{\alpha_s = \frac{g^2}{4\pi}}{=} \frac{\alpha_s^2 \pi}{s^2} \frac{1}{t^2} \frac{N_c^2 - 1}{2N_c^2} (s^2 + u^2)$$

$$\Rightarrow \frac{d\sigma}{dt}^{f + f' \to f + f'} = \frac{\pi \alpha_s^2}{s^2} \frac{N_c^2 - 1}{2N_c^2} \frac{s^2 + u^2}{t^2}$$

Heavy Quark Potential

Imagine two very heavy quarks in vacuum. Can we calculate the force one of them applies on another one? (assuming $Q\bar{Q}$ are in a color-singlet state)

In E&M one has Coulomb potential $V(r) \sim -\frac{\alpha_{EM}}{r}$.

Is it the same in QCD?

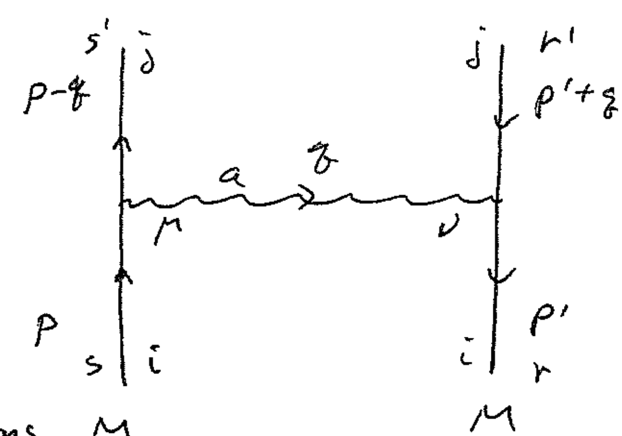
Short Distances

$Q \bullet \xleftrightarrow{r} \bullet \bar{Q}$

at small r the coupling $\alpha_s(1/r^2)$ is small

$\Rightarrow$ can do perturbation theory.

at the lowest order the potential is given by this graph:



The amplitude:

$$iM = \bar{u}_{s'}(p-q)\,\gamma^\mu\, u_s(p) \cdot$$

fermion contractions (p.122 in Peskin)

$$\cdot\, \underbrace{\bar{v}_r(p')\,\gamma^\nu\, v_{r'}(p'+q)}_{\text{need for potential}}\,(-)(ig)^2\,\underbrace{(t^a)_{ji}(t^a)_{ij}}_{\text{color singlet}}\,\frac{-i}{q^2+i\varepsilon}\,\underbrace{g_{\mu\nu}}_{\text{covariant gauge}}$$

$$\otimes \frac{d^3 q\, e^{i\vec{q}\cdot\vec{r}}}{(2\pi)^3} \cdot \frac{1}{N_c} \sim \text{average over colors (for potential only)}$$

Quark mass M is very large $\Rightarrow$

$$(p-q)^2 = M^2 \Rightarrow M^2 - 2\,p\cdot q + q^2 = M^2$$

$$\Rightarrow p\cdot q \simeq M\cdot q^0 \Rightarrow M^2 - 2M\cdot q^0 = M^2$$

$$(q^2 \ll p\cdot q) \Rightarrow q^0 = 0 \Rightarrow q^2 = -|\vec{q}|^2.$$

static case

$$\bar{u}_{s'}(p-q)\gamma^\mu u_s(p) \simeq g^{\mu 0}\cdot \bar{u}_{s'}(p-q)\gamma^0 u_s(p)$$

$$= g^{\mu 0} u^+_{s'}(p-q)\,u_s(p) = g^{\mu 0}\cdot 2M\,\delta^{ss'}$$

similarly $\bar{v}_r(p')\gamma^\nu v_{r'}(p'+q) = g^{\nu 0}\,2M\,\delta^{rr'}$

$$\cancel{i}M = +\cancel{i}g^2 \int \frac{d^3q}{(2\pi)^3}\cdot \underbrace{(2M)^2\delta^{ss'}\delta^{rr'}}_{\text{norm}}\ \frac{1}{\vec{q}^2}\ \underbrace{\mathrm{tr}(t^a t^a)\cdot\frac{1}{N_c}}_{\frac{N_c^2-1}{2N_c}\equiv C_F}$$

To get the potential need to turn d^3q into Fourier transform. Fixing the normalization write ($M \sim -V(q)$)

choose $\vec{r} = r\hat{z}$ in polar coord's

$$\boxed{V(r) = -g^2 C_F \int \frac{d^3q}{(2\pi)^3}\ \frac{e^{i\vec{q}\cdot\vec{r}}}{\vec{q}^2}} = -g^2 C_F \int_0^\infty \frac{\cancel{q^2}\,dq}{(2\pi)^3}\cdot$$

$$\cdot\int_{-1}^{1} d\cos\theta \underbrace{\int_0^{2\pi} d\varphi}_{2\pi}\ \frac{1}{\cancel{q^2}}\, e^{iqr\cos\theta} = -g^2 \frac{C_F}{(2\pi)^2}\int_0^\infty dq\,\cdot$$

$$\cdot\ \frac{1}{iqr}\left(e^{iqr} - e^{-iqr}\right) = -\frac{g^2 C_F}{4\pi^2}\frac{1}{ir}\int_0^\infty \frac{dq}{q}\left(e^{iqr} - e^{-iqr}\right)$$

$$= -\frac{g^2}{4\pi^2} C_F \frac{1}{ir} \frac{1}{2} \int_{-\infty}^{\infty} \frac{dq}{q+i\varepsilon} \left(e^{iqr} - e^{-iqr} \right)$$

close in upper half-plane | close in l.h. plane $\Rightarrow$ zero

$$= -\frac{g^2}{4\pi^{\cancel{2}}} \cdot \frac{C_F}{\cancel{i}r} \cdot \frac{1}{\cancel{2}} \cdot \cancel{2\pi i} = \Bigg| \alpha_s \equiv \frac{g^2}{4\pi} = -\frac{\alpha_s C_F}{r}$$

$$\Rightarrow \boxed{V_{QCD}(r)\Big|_{r\Lambda \ll 1} \simeq -\frac{\alpha_s C_F}{r}}$$

$\Rightarrow$ attractive Coulomb potential! just like in QED

$$\Rightarrow C_F = \frac{N_c^2 - 1}{2N_c} = \frac{8}{2 \cdot 3} = \frac{4}{3}$$

$$\Rightarrow \boxed{V_{QCD}(r)\Big|_{r\Lambda \ll 1} \simeq -\frac{4}{3}\frac{\alpha_s}{r}}.$$

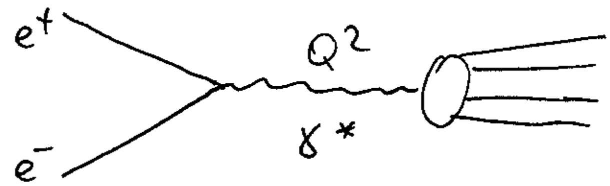
$\Rightarrow$ if one drops color factor of $4/3$ and replaces $\alpha_s \to \alpha_{EM}$ $\Rightarrow$ get QED Coulomb potential

$$V_{QED}(r) = -\frac{\alpha_{EM}}{r}.$$

The Cross Section for $e^+e^- \to$ hadrons

$\Rightarrow$ consider e^+e^- annihilation:

$$e^+e^- \to \begin{pmatrix}\text{virtual}\\ \text{photon}\end{pmatrix} \to \text{hadrons}$$



Define the ratio $R(Q^2) = \dfrac{\sigma_{e^+e^- \to \text{hadrons}}}{\sigma_{e^+e^- \to \mu^+\mu^-}}$

$R(Q^2)$ is dimensionless $\Rightarrow R = R\left(\frac{Q^2}{\mu^2}, \alpha_\mu\right)$

if $m_f = 0$. $\Rightarrow R = R\left(\frac{Q^2}{\mu^2}, \alpha_\mu\right) = (\text{put } \mu = Q) =$

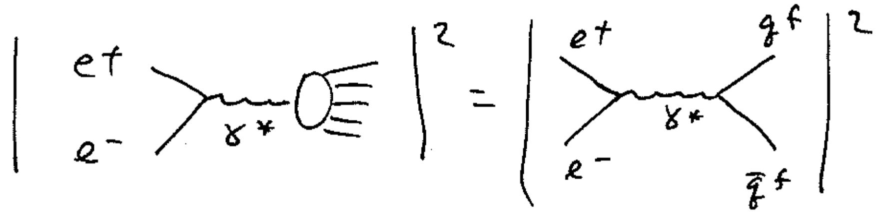
$= R(1, \alpha(Q^2)) = R(\alpha(Q^2))$ ~ function of r.c. only

$\Rightarrow$ write a perturbative expansion for it:

$$R(\alpha(Q^2)) = R(0) + R_1\,\alpha(Q^2) + R_2\,\alpha^2(Q^2) + \ldots$$

$R(0)$ is easy to get: put $\alpha(Q^2) = 0$.

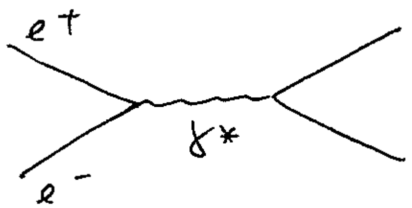
$\sigma_{e^+e^- \to \text{hadrons}} \propto$



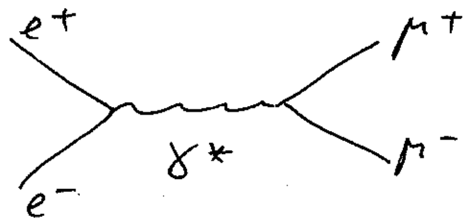
+ higher order QCD corrections

$\Rightarrow$ if $\alpha_s = 0$ $\Rightarrow$ drop higher order corrections

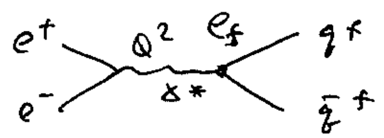
$$\Rightarrow \quad \sigma_{e^+e^- \to \text{hadrons}} \simeq \sigma_{e^+e^- \to \text{quarks}} \propto$$

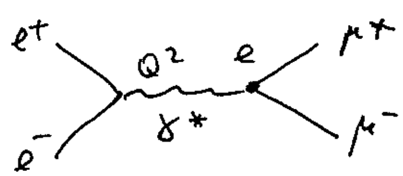
$\propto \Big|$  $\Big|^2$

On the other hand, with high precision

$\sigma_{e^+e^- \to \mu^+\mu^-} \propto \Big|$  $\Big|^2$

$\Rightarrow R(0) = \dfrac{\Big|\text{diagram } e^+e^- \to q_f \bar{q}_f\Big|^2}{\Big|\text{diagram } e^+e^- \to \mu^+\mu^-\Big|^2} = 3 \sum_f e_f^2$





neglect q & μ masses.

3 = # of quark colors

Where to terminate the sum over flavors depends on Q^2: if $Q^2 < 4m_c^2 \Rightarrow Q < 2m_c \simeq 3\,\text{GeV}$

$\Rightarrow$ need only u, d, s (3 flavors)

$$\Rightarrow R(Q < 2m_c,\ Q > 2m_s) = 3\Big(\underbrace{\left(\tfrac{2}{3}\right)^2}_{u} + \underbrace{\left(\tfrac{1}{3}\right)^2}_{d} + \underbrace{\left(\tfrac{1}{3}\right)^2}_{s}\Big) =$$

$$= 3\left(\frac{4}{9} + \frac{1}{9} + \frac{1}{9}\right) = 2$$

take $Q > 2m_b \simeq 8.5\,\text{GeV}$ $\Rightarrow$ e.g. $Q \simeq 80\,\text{GeV}$

$$\Rightarrow R = 3\Big(\underbrace{\left(\tfrac{2}{3}\right)^2}_{u} + \underbrace{\left(\tfrac{1}{3}\right)^2}_{d} + \underbrace{\left(\tfrac{1}{3}\right)^2}_{s} + \underbrace{\left(\tfrac{2}{3}\right)^2}_{c} + \underbrace{\left(\tfrac{1}{3}\right)^2}_{b}\Big) = \frac{11}{3}$$

$\Rightarrow$ amazingly close to data (see attachement)

$\Rightarrow$ if one includes higher order corrections get

$$R(\alpha(Q^2)) = 3 \sum e_f^2 \left\{ 1 + \frac{\alpha(Q^2)}{\pi} + (1.986 - 0.115 N_f) \cdot \left(\frac{\alpha}{\pi}\right)^2 + \ldots \right\}$$

$\Rightarrow$ in reality quarks become hadrons, which is a non-perturbative process ...

$\Rightarrow$ $e^+e^- \to$ hadrons gives direct evidence for quarks as fermions with 3 colors and fractional electric charges

Feynman Rules in QCD

$$\boxed{\mathcal{L}_{QCD} = \bar{q}^f\left[i\gamma^\mu D_\mu - m_f\right]q^f - \frac{1}{4}F^a_{\mu\nu}F^{a\mu\nu}}$$

However, this Lagrangian is gauge-invariant

$$\begin{cases} A_\mu \to S A_\mu S^{-1} - \frac{i}{g}(\partial_\mu S)S^{-1} \\ q \to S q \end{cases}$$

$\Rightarrow$ need to fix the gauge!

(i) Covariant (Lorenz) gauge $\partial_\mu A^{a\mu} = 0$

$\Rightarrow$ to fix the gauge need to introduce the so-called ghost fields:

$$\boxed{\mathcal{L}_{QCD}^{\text{cov. gauge}} = \bar{q}^f\left[i\gamma^\mu D_\mu - m_f\right]q^f - \frac{1}{4}F^a_{\mu\nu}F^{a\mu\nu} - \frac{1}{2\xi}(\partial_\mu A^{a\mu})(\partial_\nu A^{a\nu}) + \partial_\mu\bar{\eta}\,\mathcal{D}^\mu\eta}$$

η^a is a scalar field ~ Faddeev-Popov ghost

η^a is an anti-commuting field (Grassmann variables) (quantized like a fermion) $\Rightarrow$ unphysical $\Rightarrow$ ghosts

$\bar{\eta}^a$ is c.c. of η

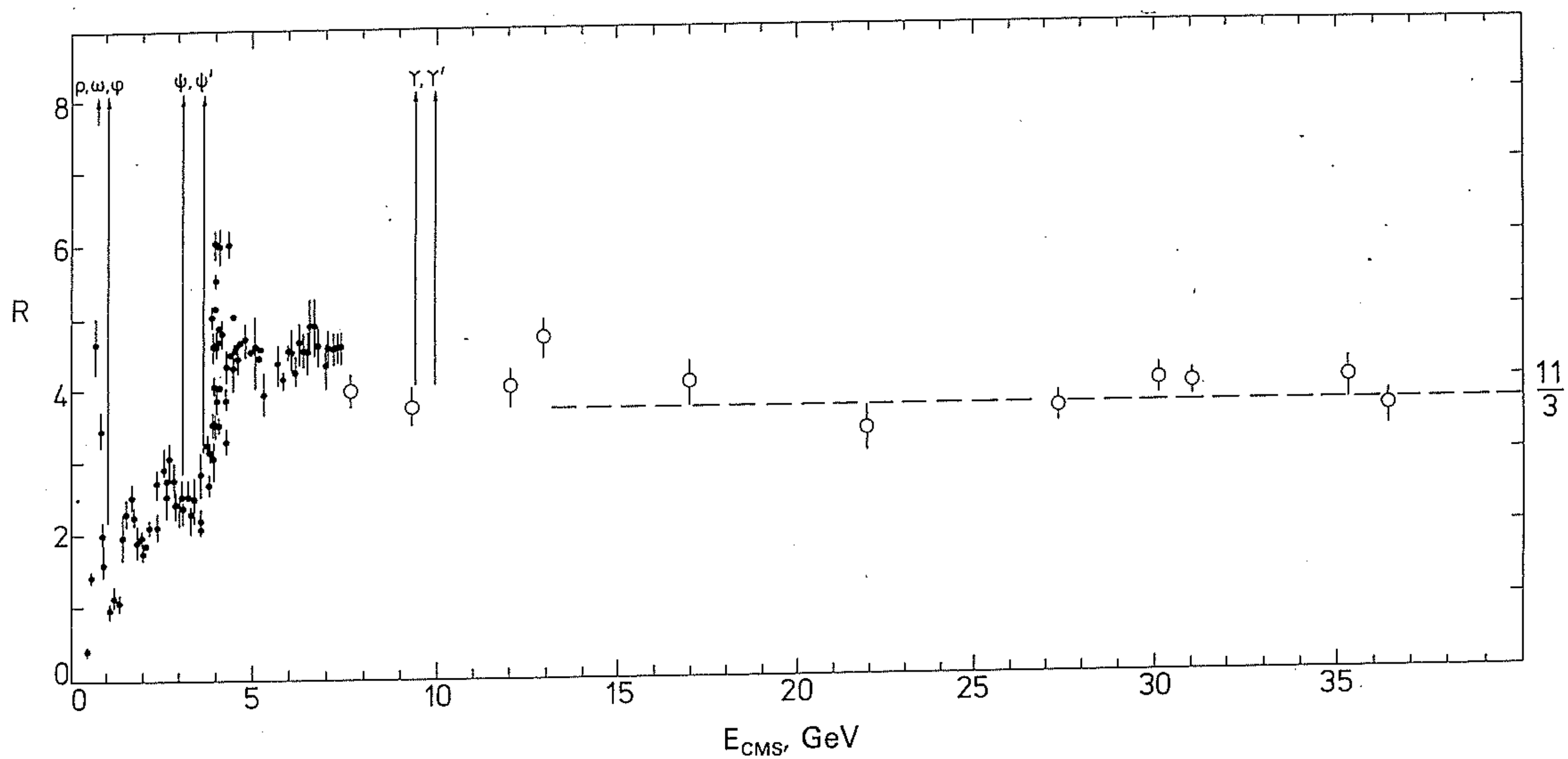


Figure 8.3 The ratio R of the cross-section for $e^+e^- \to$ hadrons, divided by that for $e^+e^- \to \mu^+\mu^-$. The fact that R is constant above 10-GeV CMS energy is a proof of the pointlike nature of hadron constituents. The predicted value of R, assuming that the primary process is formation of a quark-antiquark pair, is $\frac{11}{3}$ if pairs of u, d, s, c, b quarks are excited and they have three color degrees of freedom. The data come from many storage-ring experiments. At high energy (>10 GeV CMS) it is from the PETRA ring at DESY, Hamburg.