Last time: Running Coupling and Asymptotic Freedom (cont’d)

\[ M(g, \mu) \rightarrow M(g^\prime, \mu) \]

\[ \text{coupling must be a function of an observable} \]
\[ \text{UV cutoff \( \mu \). (need to rearrange pert. theory)} \]

\[
\left[ \frac{\mu^2}{2} \frac{\partial}{\partial \mu^2} + \beta(\mu) \frac{\partial}{\partial \mu^2} \right] M \left( \frac{Q^2}{\mu^2}, \mu^2 \right) = 0
\]

Callan-Symanzik equation \( \mu^2 \)-independence of \( M \)

**Def.**

\[ \beta(\mu^2) = \mu^2 \frac{d\beta}{d\mu^2} \]

\( \beta \)-function of a field \( \phi \)

\[ \mu^2 = \frac{g^2}{4\pi} \]

**Def.** Running coupling:

\[ \alpha(Q^2) = g^{-1} \left( \frac{\hbar}{\mu^2} + \beta(\mu) \right) \]

where \[ \beta(\mu) = \int_{\mu_0}^{\mu} \frac{d\mu'}{\beta(\mu')} \]

\( \alpha(Q^2) \) is \( \mu^2 \)-independent.

\[ M \left( \frac{Q^2}{\mu^2}, \mu^2 \right) = M \left( 1, \alpha(Q^2) \right) = M(\alpha(Q^2)) \] is also
$\mu^2$-independent

Any ftn. of $2(\Omega^4)$ is $\mu^2$-independent.
Last time, I finished talking about the running coupling. In QCD we observed that the 1-loop running coupling is

\[\alpha_s(Q^2) = \frac{d\alpha}{d\ln Q^2} \]

\[\beta_2 = \frac{11N_c - 2N_f}{12\pi} \]

\[\alpha_s(Q^2) = \frac{1}{\beta_2 \beta_0 \frac{Q^2}{\Lambda_{\text{QCD}}^2}}\]

\[Q^2 = \Lambda_{\text{QCD}}^2 \sim \text{Landau pole}\]

In QED:

\[\alpha_{\text{EM}}(Q^2) = \frac{\alpha}{1 - \frac{\alpha}{3\pi} \ln \frac{Q^2}{\mu^2}}\]

Note the sign.
\[ d_{EM}(Q^2) = \frac{d\mu}{1 + \frac{d\mu}{\beta_2^2} \ln \frac{Q^2}{\mu^2}} = \frac{d\mu}{1 - \frac{d\mu}{3 \pi} \ln \frac{Q^2}{\mu^2}} \]

\[ \Rightarrow d_{EM}(Q^2) = \frac{\frac{d\mu}{1 + \frac{d\mu}{3 \pi} \ln \frac{m^2}{Q^2}}} \]

\[ \text{n increases with } Q^2 \]

\[ d_{EM}(Q^2) \]

\[ \sim \frac{1}{137} \]

\[ \sim \frac{1}{128} \]

\[ \text{Jackson scale} \]

\[ \text{Landau pole} \]

\[ \text{Landau pole} \]

\[ \Rightarrow \text{no asymptotic freedom in QED!} \]

\[ \Rightarrow \text{also has a Landau pole, but at large momenta in QED may map onto some more "fundamental" theory, eliminating Landau pole} \ldots \]
\( \Rightarrow \) in QCD with massless quarks mesons are massless.

\( \Rightarrow \) baryons have a mass. Consider proton (the lightest baryon).

proton mass: \( M_p \) a dimensionfull quantity.

\( M_p = M_p (\beta, \mu) = \mu f(\beta, \mu) \) as \( \mu \) is the only dimension full scale.

\( \mu^2 \frac{d}{d\mu^2} M_p = 0 \Rightarrow \left( \mu^2 \frac{2}{\mu^2} + \beta(\beta, \mu) \frac{2}{d\beta} \right) M_p = 0 \)

\( \left( \mu^2 \frac{2}{\mu^2} + \beta(\beta, \mu) \frac{2}{d\beta} \right) \left[ \mu f(\beta, \mu) \right] = 0 \)

\( \mu^2 \frac{2}{\mu^2} (\beta, \mu) = \frac{1}{2} \beta(\beta, \mu) \Rightarrow \left( \frac{1}{2} + \beta(\beta, \mu) \right) f(\beta, \mu) = 0 \)

\( \Rightarrow \frac{df(\beta, \mu)}{d\beta} = -\frac{1}{2 \beta(\beta, \mu)} f(\beta, \mu) = -\frac{df}{f} = -\frac{d\beta}{2 \beta(\beta, \mu)} \)

\( \Rightarrow \ln f(\beta, \mu) - \ln f(\beta, \mu_0) = -\frac{1}{2} \int_{\beta_0}^{\beta} \frac{d\beta'}{\beta(\beta', \mu_0)} = -\frac{1}{2} \beta(\beta, \mu_0) \)

\( \Rightarrow f(\beta, \mu) = f(\beta, \mu_0) e^{-\frac{1}{2} \beta(\beta, \mu_0)} \) and the proton's mass is
\[ M_p = M \cdot f(x_0) \cdot e^{-\frac{1}{2} \int p(x_0, x_0)} \]

Take \( p(x) = \frac{dx}{\beta_2(x')} = \frac{1}{\beta_2} \left( \frac{1}{x'} - \frac{1}{x_0} \right) \)

\[ \Rightarrow M_p = \mu \cdot f(x_0) \cdot e^{-\frac{1}{2\beta_2} \left( \frac{1}{x'} - \frac{1}{x_0} \right)} \]

\( M_p \) should not depend on \( x_0 \) (a cut-off) \( \Rightarrow \)

\[ \Rightarrow f(x_0) \propto e^{-\frac{1}{2\beta_2} \frac{1}{x_0}} \Rightarrow \text{write } f(x_0) = C_p e^{-\frac{1}{2\beta_2} \frac{1}{x_0}} \]

\[ \Rightarrow \begin{aligned} M_p &= C_p \cdot \mu \cdot e^{-\frac{1}{2\beta_2} \frac{1}{x'}} \\ &\text{non-perturbative dependence on } x' \\ e^{-\frac{1}{x'}} \text{ is a function } \neq \text{ to its Taylor series} \Rightarrow \text{non-perturbative!} \end{aligned} \]

Take \( p(x) = -\beta_2 x^2 - \beta_3 x^3 \Rightarrow \)

\[ \begin{aligned} p(x) &= -\frac{1}{\beta_2} \int_{x_0}^{x'} \frac{dx'}{\beta_2 (1 + \beta_3 \frac{x'}{\beta_2})} \\ &= -\frac{1}{\beta_2} \int_{x_0}^{x'} \frac{dx'}{\beta_2} \left[ 1 - \frac{\beta_3}{\beta_2} x' + \ldots \right] \\ &= \frac{1}{\beta_2} \left( \frac{1}{x'} - \frac{1}{x_0} \right) + \frac{\beta_3}{\beta_2^2} \ln \frac{x'}{x_0} + \ldots \\ \Rightarrow M_p &= \mu \cdot f(x_0) \cdot e^{-\frac{1}{2} \left[ \frac{\beta_2}{\beta_2} \left( \frac{1}{x'} - \frac{1}{x_0} \right) + \frac{\beta_3}{\beta_2^2} \ln \left( \frac{x'}{x_0} \right) + \ldots \right]} \end{aligned} \]
\[ f(x_0) = \exp \left( \frac{-1}{2\beta_2 x_0} + \frac{\beta_3}{2\beta_2^2} \ln x_0 \right) \]

\[ M_p = C_p \mu e^{-\frac{1}{2\beta_2 \lambda g_0}} \left( \lambda g_0 \right)^{\frac{\beta_3}{2\beta_2^2}} \left( 1 + o(\lambda g_0) \right) \]

Non-analytic function.

Analytic function.

\[ \Rightarrow \text{can not calculate } M_p \text{ in perturbation theory.} \]

Finally, 
\[ M_p = C_p \mu e^{-\frac{1}{2\beta_2 \lambda g_0}} \]

Remember that 
\[ \lambda g_0 = \frac{1}{\beta_2 \ln \frac{\mu^2}{\Lambda_{QCD}^2}} \Rightarrow \frac{1}{2\beta_2 \lambda g_0} = \ln \frac{\mu}{\Lambda_{QCD}} \]

\[ M_p = C_p \mu e^{-\ln \frac{\mu}{\Lambda_{QCD}}} = C_p \Lambda_{QCD} \]

\[ \Rightarrow (M_p \sim \Lambda_{QCD}) \]

\[ \text{in a non-perturbative QCD scale where the coupling } \lambda g_0 \]

\[ \text{is large } \Rightarrow \text{can't do perturbation theory there.} \]
\[ W(x, y) = \frac{\partial}{\partial x_i} \left( \prod \delta(x) \right) \frac{\partial}{\partial x_i} \left( \prod \delta(x) \right) \]
\[
\prod_{i=1}^{N} \left[ 1 + i g \left( \Delta x_i \right) A_\mu (x_i) \right] S^{-1} (x_i) = \prod_{i=1}^{N} \left[ \mathcal{A} + i g \left( \Delta x_i \right) A_\mu (x_i) S^{-1} (x_i) \right] = \prod_{i=1}^{N} S^{-1} (x_{i-1}).
\]

\[
\left[ 1 + i g \Delta x \right] S^{-1} (x) = S (x) \prod_{i=1}^{N} \left[ 1 + i g \Delta x^\mu A_\mu (x_i) \right] S^{-1} (y).
\]

\[
W_c (x, y) \rightarrow S (x) W_c (x, y) S^{-1} (y).
\]

**Def.** Wilson loop:

\[ \text{tr} [W_c (x, x)] \text{ is called a } \text{Wilson loop}. \]

(K. Wilson, '74?)

Under gauge transformation,

\[ \text{tr} [W_c (x, x)] \rightarrow \text{tr} [S (x) W_c (x, x) S^{-1} (x)] = \text{tr} [W_c (x, x)] \]

invariant! Wilson loop is gauge-invariant!
Wilson line represents quark propagator when one can neglect recoil. This works in high energy scattering and for static heavy quarks.

Wilson lines form links which can be used to define QCD action on the lattice for numerical simulations.

**Heavy Quark Potential:**

Suppose one wants to find heavy $Q\bar{Q}$ potential in QCD. How does one define the potential $V(r)$ in a gauge-invariant way?

Take a Wilson loop defined as shown.

$$\langle W \rangle \bigg|_{T \to \infty} = e^{-i T V(r)}$$

neglect interaction with gauge links (it does not scale with $T$ to the same degree)

$$V(r) = \lim_{T \to \infty} \left[ \frac{i}{T} \ln \langle W \rangle \right]$$

Can calculate numerically on the lattice.
Note that, since Feynman path integral time-ordered operators, one can write

\[
\text{tr} [ W_c(x,x) ] = \frac{1}{\mathcal{Z}} \int \mathcal{D}A_\mu \ e^{ig \int j^a_\mu(x) A^a_\mu(x) d^4x} e^{i S[A_\mu]}
\]

where \( j^a_\mu(x) \) is some external current, which is non-zero only along the contour \( C \).

\[ R \] is the only scale in \( V(r) \Rightarrow \delta_S = \delta_S \left( \frac{1}{r^2} \right) \]

if \( R \ll \frac{1}{\Lambda_{\text{QCD}}} \Rightarrow \delta_S \left( \frac{1}{r^2} \right) \ll 1 \Rightarrow \) can use perturbative QCD

The potential is (see pp. 38-40 of these notes)

\[
V(r) \bigg|_{r \ll \frac{1}{\Lambda_{\text{QCD}}}} \sim - \frac{\alpha_S C_F}{r} = - \frac{4}{3} \frac{\alpha_S}{r}
\]

\( \Rightarrow \) this is a Coulomb-like potential, similar to classical E&M.