Last time | finished talking about the Parton model

$\Rightarrow$ derived Callan-Gross relation:

$$\boxed{F_2(x) = 2x\, F_1(x)}$$

note: this is specific to spin-$\frac{1}{2}$ quarks
(it would be different for particles of diff. spin)

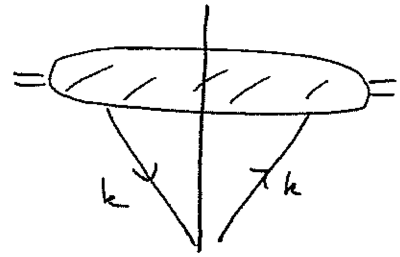
Defined quark distribution function:

$$\boxed{q^f(x) \equiv \frac{1}{2p^+} \int \frac{d^4k}{(2\pi)^4} A^f_{\alpha\beta}(p,k)\,(\gamma^+)_{\beta\alpha}\, \delta\left(x - \frac{k^+}{p^+}\right)}$$

Obtained

$$\boxed{\begin{aligned} F_2(x) &= \sum_f e_f^2\, x\, q^f(x) \\ F_1(x) &= \frac{1}{2}\sum_f e_f^2\, q^f(x) \end{aligned}}$$

F_1, F_2 are independent of Q^2, and depend only on x:
this is Bjorken scaling.

$q^f(x) = \dfrac{1}{2p^+}$  $\Rightarrow$ often $p^f(x)$ is denoted $q(x)$.

$\nwarrow \gamma^+ \delta\left(x - \frac{k^+}{p^+}\right)$

> $q^f(x, Q^2)$ counts # of quarks with light cone momentum x and transverse momentum $k_T \leq Q$.

parton distribution function $\left(q^f \sim a^+ a \right)$

$\Rightarrow$ for a free quark $A^f_{ab}(p,k)(\gamma^+)_{ba} = \delta^4(p-k) \cdot (2\pi)^4$.

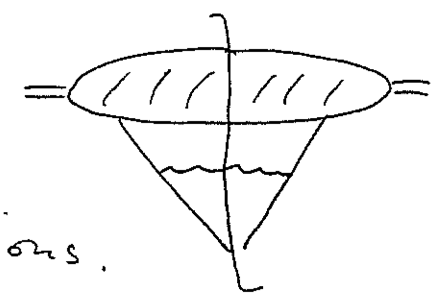
$$\underbrace{\bar{u}_b(p)(\gamma^+)_{ba} u_a(p)}_{= 2p^+} = 2p^+ (2\pi)^4 \delta^4(p-k) \Rightarrow \boxed{q^f_{\text{quark}}(x) = \delta(x-1)}$$

plug in.

one quark at $x=1$

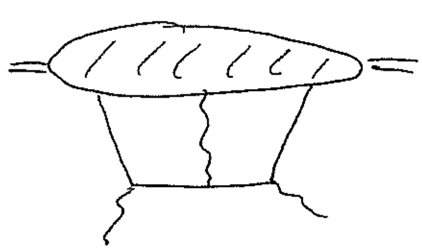
Peskin, ch. 17.5
Sterman ch 14
K, Levin ch. 2.4

QCD Improved Parton Model: DGLAP equation

How about corrections like  ?

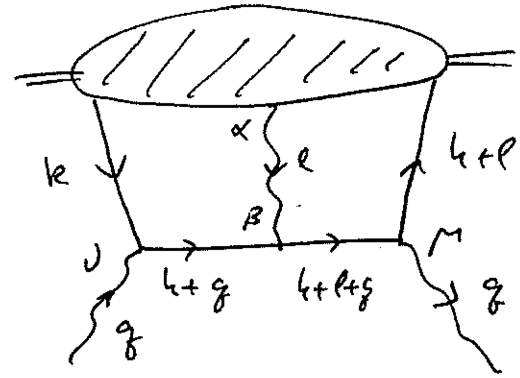
These are important corrections.

However, let us first discard the negligible diagrams like



Work in Light Cone (LC) gauge: $\eta \cdot A = A^+ = 0$

$(\eta^+ = 0, \eta^- = 1, \underline{\eta} = 0)$



$Q^2 \sim$ very large:

$$\Gamma_{\mu\nu\beta} = \gamma_\mu \frac{\gamma\cdot(k+\ell+q)}{(k+\ell+q)^2 + i\varepsilon} \gamma_\beta \cdot \frac{\gamma\cdot(k+q)}{(k+q)^2+i\varepsilon}\gamma_\nu$$

; Now, $(q+k)^2 \simeq q^2 + 2k\cdot q = -Q^2 + 2k^+ q^-$

as k^+ is large. Similarly $(k+\ell+q)^2 \simeq -Q^2 + 2q^-(k^+ + \ell^+)$

$$\Rightarrow \Gamma_{\mu\nu\beta} = \frac{1}{Q^4} \frac{\gamma_\mu\, \gamma\cdot(k+\ell+q)\, \gamma_\beta\, \gamma\cdot(k+q)\, \gamma_\nu}{\left(1 - \frac{2k^+q^-}{Q^2} - i\varepsilon\right)\left(1 - \frac{2(k^+ + \ell^+)q^-}{Q^2} - i\varepsilon\right)}$$

Seems like $\Gamma_{\mu\nu\beta} = O\left(\frac{1}{Q^4}\right)$, which is suppressed compared to $O\left(\frac{1}{Q^2}\right)$ diagram

However, when integrating over $d\ell^+$ may pick up the pole at $\ell^+ = -k^+ + \frac{Q^2}{2q^-}$, getting a Q^2 in the numerator.

$$\gamma\cdot(k+q) \simeq \gamma^+(k^-+q^-) + \gamma^-(k^++q^+) - \underline{\gamma}\cdot(\underline{k}+\underline{q})$$

$$\gamma\cdot(k+l+q) = \gamma^+(k^-+l^-+q^-) + \gamma^-(k^++l^++q^+) - \underline{\gamma}\cdot(\underline{k}+\underline{l}+\underline{q})$$

(or integrating over k^+ picking up a pole)

$\Rightarrow$ When taking Im part get $2k^+q^- = Q^2 \Rightarrow$

$\Rightarrow q^- = \dfrac{Q^2}{2k^+} \Rightarrow$ 2nd denominator becomes

Q^2-independent: $1 - \dfrac{2(k^++l^+)q^-}{Q^2} = 1 - \dfrac{k^++l^+}{k^+} = -\dfrac{l^+}{k^+}$.

$\Rightarrow$ keep only q^- terms in the numerator (Q^2-dep.)

$\Rightarrow \gamma\cdot(k+q) \simeq \gamma^+ q^-$, $\gamma\cdot(k+l+q) \simeq \gamma^+ q^-$

$\Rightarrow \Gamma_{\mu\nu\beta} \sim \gamma^\mu \gamma^+ \gamma^\beta \gamma^+ \gamma^\nu$

$\Rightarrow \gamma^+\gamma^\beta\gamma^+$ is $\neq$ only if $\beta =$ "$-$" as $(\gamma^+)^2 =$

$$= \left(\frac{\gamma^0+\gamma^3}{\sqrt{2}}\right)^2 = \frac{1}{2}\left((\gamma^0)^2 + (\gamma^3)^2 + \{\gamma^0,\gamma^3\}^{=0}\right) = \frac{1}{2}(1-1) = 0.$$

But if $\beta = -$ $\Rightarrow$ need $D^{\alpha +}(l)$

$$D_{\alpha\beta}(l) = \frac{-i}{l^2}\left[g_{\alpha\beta} - \frac{\eta_\alpha l_\beta + \eta_\beta l_\alpha}{\eta\cdot l}\right]$$

$$D^{\alpha +} = \frac{-i}{l^2}\left[g^{\alpha +} - \frac{\eta^\alpha l^+}{l^+}\right] = 0$$ $\Rightarrow$ never get Q^2

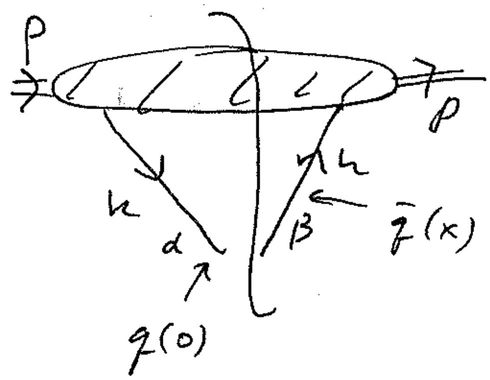
in the numerator $\Rightarrow$ $O\left(\dfrac{1}{Q^4}\right)$ "Higher Twist"

Operator Definition of Quark Distribution

Start with

$$q^f(x_{Bj}) = \frac{1}{2p^+}\int \frac{d^4k}{(2\pi)^4} A^f_{\alpha\beta}(p,k)\left(\gamma^+\right)_{\beta\alpha}\, \delta\!\left(x - \frac{k^+}{p^+}\right)$$

where $A^f_{\alpha\beta}(p,k) =$



Note that

$$A^f_{\alpha\beta}(p,k)\left(\gamma^+\right)_{\beta\alpha} = \mathrm{tr}\left[\gamma^+ A^f(p,k)\right] =$$

$$= \int d^4x\, e^{ix\cdot k} \langle p|\, \bar{q}(x)\, \gamma^+ q(0)\, |p\rangle$$

proton state

quark field operators

$$\Rightarrow q^f(x_{Bj}) = \frac{1}{2p^+}\int \frac{d^4k}{(2\pi)^4} d^4x\, e^{ix\cdot k} \langle p|\bar{q}(x)\gamma^+ q(0)|p\rangle$$

$$\cdot\, \delta\!\left(x_{Bj} - \frac{k^+}{p^+}\right) = \frac{1}{4\pi}\int_{-\infty}^{\infty} dx^-\, e^{i x_{Bj} p^+ x^-} \langle p|\bar{q}(x)\gamma^+ q(0)|p\rangle$$

Under local SU(3) gauge transformations

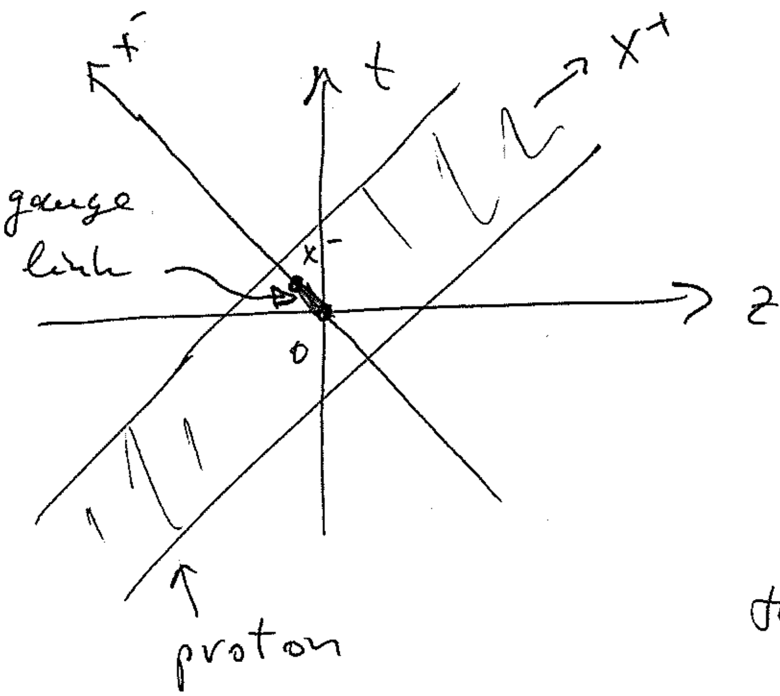
$$\begin{cases} q(x) \to S(x)\, q(x) \\ \bar{q}(0) \to \bar{q}(0)\, S^+(0) \end{cases}$$

$\Rightarrow$ while $S^+(x)\, S(x) = \mathbb{1} = S^+(0)\, S(0)$,

$S^+(x)\, S(0) \neq \mathbb{1} \Rightarrow \bar{q}(x)\, \gamma^+ q(0)$ is not gauge-invariant!

Introduce a gauge link, a Wilson line, connecting 0 to x, to make our definition gauge-invariant (cf. Eq. (14.536) in Sterman)

$$\boxed{q(x_{Bj}, Q^2) = \frac{1}{4\pi} \int\limits_{-\infty}^{\infty} dx^- \, e^{i x_{Bj} p^+ x^-} \cdot \langle p | \bar{q}(x)\, \gamma^+ \cdot P \exp\left\{ ig \int\limits_0^{x^-} dx'^- A^+(0, x'^-, \underline{0}) \right\} \cdot q(0) | p \rangle}$$



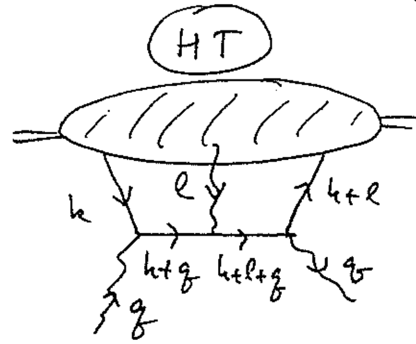
The contour is chosen to mimic the quark propagator in DIS.

In $A^+ = 0$ gauge

$P \exp\left\{ ig \int dx'^- A^+ \right\} = \mathbb{1}$,

the link disappears!

Let us work in Light Cone gauge defined by



$$\eta \cdot A = A^+ = 0$$

$$(\eta^+ = 0,\ \eta^- = 1,\ \underline{\eta} = 0)$$

In DIS Q^2 is very large such that

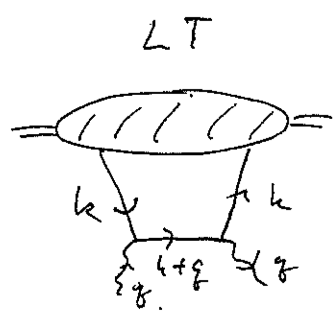
$$\frac{|k^2|}{Q^2} \ll 1, \quad \frac{|l^2|}{Q^2} \ll 1$$

$\Rightarrow$ approximate $(k+q)^2 \simeq q^2 \simeq -Q^2$

$$(k+l+q)^2 \simeq q^2 \simeq -Q^2$$

$\Rightarrow$ diagram HT $\sim \frac{1}{Q^4}$

Compare with leading parton model diagram LT $\sim \frac{1}{Q^2}$



$\Rightarrow$ HT $\ll$ LT as for large Q^2: $\frac{1}{Q^4} \ll \frac{1}{Q^2}$.

HT stand for Higher Twist $\sim 1/Q^4$

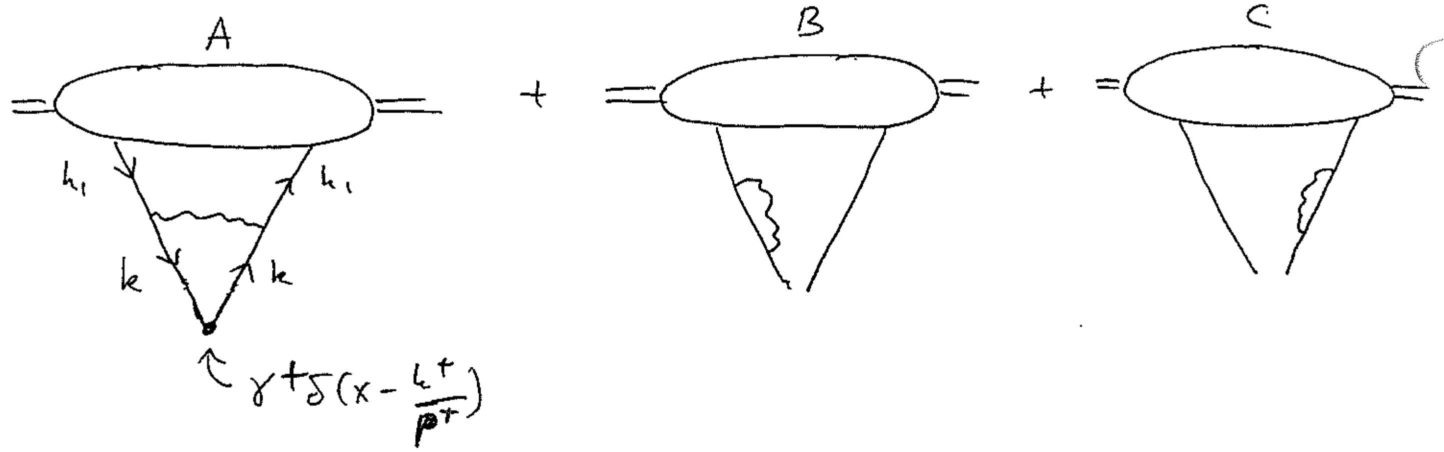
LT —"— Leading Twist $\sim 1/Q^2$

$\Rightarrow$ Multiple rescatterings are <u>Higher Twists</u>,

usually suppressed by Λ^2/Q^2 (Λ - some small scale)

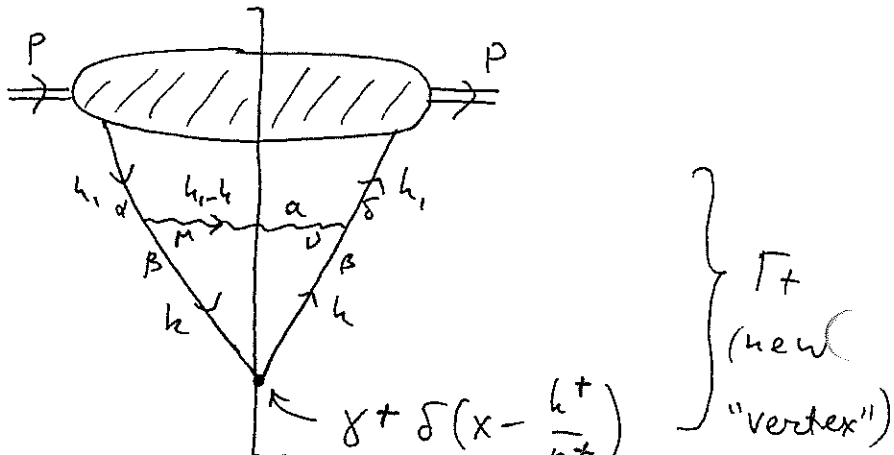
(true in LC gauge only!)

We need to calculate the following corrections to the parton model:



We will work out diagram A only: $|\underline{k}| \gg |\underline{k}_1|$

$$q^f(x,Q^2) = \frac{1}{2p^+}$$



$$(t^a t^a)_{\delta\alpha} = \delta_{\alpha\delta}\frac{N_c^2-1}{2N_c}$$

$$\Gamma^+ = (ig)^2\,\overbrace{(t^a)_{\delta\beta}\,(t^a)_{\beta\alpha}}\int\frac{d^4k}{(2\pi)^4}\,\gamma^\nu\,\frac{i\gamma\cdot k}{k^2}\,\gamma^+\,\frac{i\gamma\cdot k}{k^2}\,\gamma^\mu\cdot$$

$$\cdot\,\delta\left(x-\frac{k^+}{p^+}\right)(-2\pi)\,\delta((k_1-k)^2)\left[g_{\mu\nu}-\frac{\eta_\mu(k_{1\nu}-k_\nu)+\eta_\nu(k_{1\mu}-k_\mu)}{k_1^+-k^+}\right]$$

where we used the fact that gluon propagator in the $\eta\cdot A = A^+ = 0$ light cone gauge is

$$D_{\mu\nu}(l) = \frac{-i}{l^2}\left[g_{\mu\nu}-\frac{\eta_\mu l_\nu+\eta_\nu l_\mu}{\eta\cdot l}\right] \quad \text{with } \frac{-i}{l^2}\to -2\pi\,\delta(l^2)$$

and $\eta\cdot l = l^+$.

First integrate over k_-:

$$\int dk^- \, \delta((k_1-k)^2) = \frac{1}{2(k_1-k)^+} \quad \text{with} \quad k^- = k_1^- - \frac{(\underline{k}_1-\underline{k})^2}{2(k_1-k)^+}$$

Also, k^+-integration is easy: $\int dk^+ \, \delta(x - \frac{k^+}{p^+}) = p^+$

Defining $\alpha_s \equiv \frac{g^2}{4\pi}$ we write $\left(C_F \equiv \frac{N_c^2-1}{2N_c}\right)$

$$\Pi^+ = -\frac{\alpha_s C_F}{4\pi^2} \delta_{\alpha\beta} \int \frac{d^2k}{k^4} \gamma^\nu \gamma\cdot k \, \gamma^+ \gamma\cdot k \, \gamma^\mu \left[g_{\mu\nu} - \frac{\eta_\mu(k_{1\nu}-k_\nu)+\eta_\nu(k_{1\mu}-k_\mu)}{k_1^+-k^+} \right] \frac{p^+}{(k_1-k)^+}$$

Evaluate k^2: $k^2 = 2k^+k^- - \underline{k}^2 = 2k^+\left(k_1^- - \frac{(\underline{k}_1-\underline{k})^2}{2(k_1-k)^+}\right) - \underline{k}^2 =$

$= \Big|$ define $z = \frac{k^+}{k_1^+}$ $= 2z k_1^+ k_1^- - \frac{z}{1-z}(\underline{k}_1-\underline{k})^2 - \underline{k}^2 =$

$$= z k_1^2 + z\underline{k}_1^2 - \frac{z}{1-z}(\underline{k}_1^2 - 2\underline{k}_1\cdot\underline{k} + \underline{k}^2) - \underline{k}^2 =$$

$$= z k_1^2 - \frac{1}{1-z}(\underline{k} - z\underline{k}_1)^2 \simeq -\frac{\underline{k}^2}{1-z} \approx k^2 \quad \text{for } k_\perp \gg k_{1\perp},\ \underline{k}^2 \gg k_1^2$$

Evaluate $\gamma^\nu \gamma\cdot k \, \gamma^+ \gamma\cdot k \, \gamma^\mu \left[g_{\mu\nu} - \frac{\eta_\mu(k_1-k)_\nu + \eta_\nu(k_1-k)_\mu}{k_1^+-k^+} \right]$

First note that as $\{\gamma_\rho, \gamma_\sigma\} = 2g_{\rho\sigma}$

$$\gamma\cdot k \, \gamma^+ \gamma\cdot k = \gamma\cdot k \left[\underbrace{\{\gamma^+, \gamma\cdot k\}}_{2k^+} - \gamma\cdot k \, \gamma^+ \right] = \underset{(2)}{2k^+ \gamma\cdot k} - \underset{(1)}{k^2\gamma^+}$$

Let's put ① and ② back into the monster expression:

$$① = -k^2 \gamma^\nu \gamma^+ \gamma^\mu \left[g_{\mu\nu} - \frac{\cancel{\eta_\mu (k_1-k)_\nu} + \cancel{\eta_\nu (k_1-k)_\mu}}{\eta\cdot(k_1-k)} \right] =$$

$$= \left(\text{as } \gamma^{+2} = 0\right) = -k^2 \gamma_\mu \gamma^+ \gamma^\mu = 2\gamma^+ k^2$$

since $\gamma_\mu \gamma_\alpha \gamma^\mu = -2\gamma_\alpha$.

$$② = 2k^+ \gamma^\nu \gamma\cdot k \gamma^\mu \left[g_{\mu\nu} - \frac{\eta_\mu (k_1-k)_\nu + \eta_\nu (k_1-k)_\mu}{\eta\cdot(k_1-k)} \right] =$$

$$= 2k^+ \left[\gamma_\mu \gamma\cdot k \gamma^\mu - \frac{1}{\eta\cdot(k_1-k)} \left(\gamma\cdot(k_1-k)\gamma\cdot k \gamma^+ + \gamma^+ \gamma\cdot k \gamma\cdot(k_1-k) \right) \right] = 2k^+ \left[-2\gamma\cdot k - \frac{1}{\eta\cdot(k_1-k)} \cdot \right.$$

$$\left. \cdot \left(-2k^2\gamma^+ + \gamma\cdot k_1 \gamma\cdot k \gamma^+ + \gamma^+ \gamma\cdot k \gamma\cdot k_1 \right) \right]$$

we want to swap and move over here.

$$② = 2k^+ \left[-2\gamma\cdot k - \frac{1}{\eta\cdot(k_1-k)} \left(-2k^2\gamma^+ + 2k\cdot k_1 \gamma^+ + \right.\right.$$

$$\left.\left. + 2k^+ \gamma\cdot k_1 - 2k_1^+ \gamma\cdot k \right) \right]$$

Now we are interested in the regime where

$|\underline{k}| \gg |\underline{k}_1|$, $\underline{k}^2 \gg k_1^2$, i.e. $|\underline{k}|$ is VERY LARGE.

at the leading order in $|\underline{k}|$: $k^- \approx -\dfrac{\underline{k}^2}{2k_1^+(1-z)}$