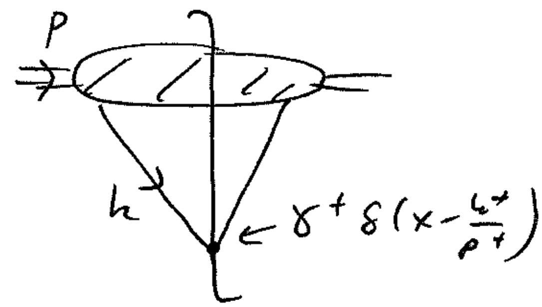
Last time | QCD-Improved Parton Model: DGLAP equation (cont'd)

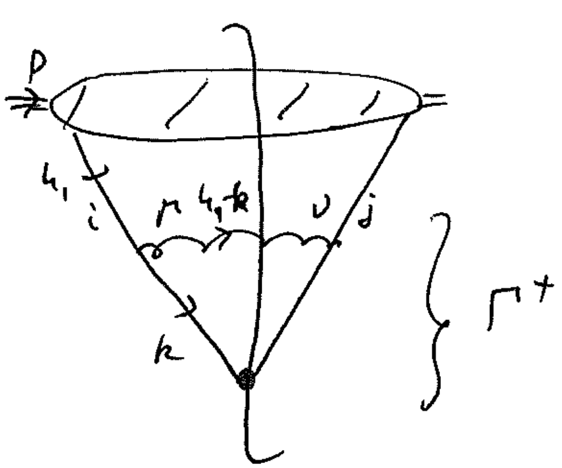
Operator definition of quark PDF:

$$\boxed{q(x,Q^2) = \frac{1}{4\pi}\int_{-\infty}^{\infty} dx^- \, e^{ixp^+x^-} \langle p | \bar{q}(x)\, \gamma^+ \cdot \mathrm{Pexp}\left\{ ig \int_0^{x^-} dx'^- A^+(0,x'^-,\underline{0}) \right\} q(0) | p \rangle}$$

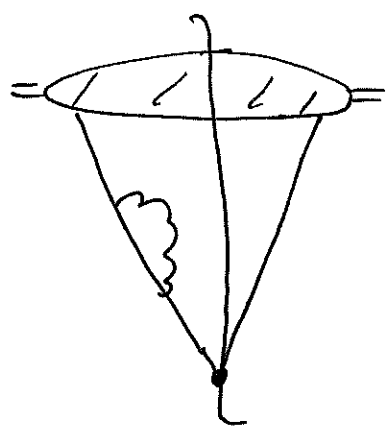
This is represented by



We need to calculate corrections, to obtain Q^2-dependence



$+$



$+$ c.c.

$$\boxed{\Gamma^+ = -\frac{\alpha_s C_F}{4\pi^2} \delta_{ij} \int \frac{d^4k}{(k^2)^2} \gamma^\nu \not{k} \gamma^+ \not{k} \gamma^\mu \left[g_{\mu\nu} - \frac{\eta_\mu (k_1-k)_\nu + \eta_\nu (k_1-k)_\mu}{\eta\cdot(k_1-k)} \right] \cdot \frac{p^+}{(k_1-k)^+}}$$

assume $Q^2 \gg k_\perp^2 \gg \underline{k}_1^2, k_1^2 \gg \Lambda_{QCD}^2$.

Then $k^2 \simeq - \frac{\underline{k}^2}{1-z}$. $z \equiv k^+/k_1^+$

moreover, $\not{k} \gamma^+ \not{k} = \underbrace{-k^2 \gamma^+}_{①} + \underbrace{2k^+ \not{k}}_{②}$

① $\simeq 2\gamma^+ k^2$ (including γ^ν, γ^μ & the gluon propagator)

First integrate over k_-:

$$\int dk^- \, \delta((k_1-k)^2) = \frac{1}{2(k_1-k)^+} \quad \text{with} \quad k^- = k_1^- - \frac{(\underline{k}_1-\underline{k})^2}{2(k_1-k)^+}$$

Also, k^+-integration is easy: $\int dk^+ \, \delta(x - \frac{k^+}{p^+}) = p^+$

Defining $\alpha_s \equiv \frac{g^2}{4\pi}$ we write $\left(C_F \equiv \frac{N_c^2-1}{2N_c}\right)$

$$\Pi^+ = -\frac{\alpha_s C_F}{4\pi^2} \delta_{\alpha\delta} \int \frac{d^2k}{k^4} \gamma^\nu \gamma\cdot k \, \gamma^+ \gamma\cdot k \, \gamma^\mu \left[g_{\mu\nu} - \frac{\eta_\mu(k_{1\nu}-k_\nu)+\eta_\nu(k_{1\mu}-k_\mu)}{k_1^+-k^+} \right] \frac{p^+}{(k_1-k)^+}$$

Evaluate k^2: $k^2 = 2k^+k^- - \underline{k}^2 = 2k^+\left(k_1^- - \frac{(\underline{k}_1-\underline{k})^2}{2(k_1-k)^+}\right) - \underline{k}^2 =$

$= \Big|$ define $z = \frac{k^+}{k_1^+}$ $= 2z\, k_1^+ k_1^- - \frac{z}{1-z}(\underline{k}_1-\underline{k})^2 - \underline{k}^2 =$

$$= z\, k_1^2 + z\, \underline{k}_1^2 - \frac{z}{1-z}\left(\underline{k}_1^2 - 2\underline{k}_1\cdot\underline{k} + \underline{k}^2\right) - \underline{k}^2 =$$

$$= z\, k_1^2 - \frac{1}{1-z}(\underline{k} - z\underline{k}_1)^2 \simeq -\frac{\underline{k}^2}{1-z} \quad \text{for } k_\perp \gg k_{1\perp},\ \underline{k}^2 \gg k_1^2$$

$\tilde{k}^2$

Evaluate $\gamma^\nu \gamma\cdot k \, \gamma^+ \gamma\cdot k \, \gamma^\mu \left[g_{\mu\nu} - \frac{\eta_\mu(k_1-k)_\nu + \eta_\nu(k_1-k)_\mu}{k_1^+-k^+} \right]$

First note that as $\{\gamma_\rho, \gamma_\sigma\} = 2g_{\rho\sigma}$

$$\gamma\cdot k\, \gamma^+ \gamma\cdot k = \gamma\cdot k \Big[\underbrace{\{\gamma^+, \gamma\cdot k\}}_{2k^+} - \gamma\cdot k\, \gamma^+ \Big] = \underset{(2)}{2k^+ \gamma\cdot k} - \underset{(1)}{k^2 \gamma^+}$$

Let's put ① and ② back into the monster expression:

$$① = -k^2 \gamma^\nu \gamma^+ \gamma^\mu \left[g_{\mu\nu} - \frac{\cancel{\eta_\mu (k_1-k)_\nu} + \cancel{\eta_\nu (k_1-k)_\mu}}{\eta\cdot(k_1-k)} \right] =$$

$$= \left(\text{as } \gamma^{+2} = 0\right) = -k^2 \gamma_\mu \gamma^+ \gamma^\mu = 2\gamma^+ k^2$$

since $\gamma_\mu \gamma_\alpha \gamma^\mu = -2\gamma_\alpha$.

$$② = 2k^+ \gamma^\nu \gamma\cdot k\, \gamma^\mu \left[g_{\mu\nu} - \frac{\eta_\mu (k_1-k)_\nu + \eta_\nu (k_1-k)_\mu}{\eta\cdot(k_1-k)} \right] =$$

$$= 2k^+ \Big[\gamma_\mu \gamma\cdot k\, \gamma^\mu - \frac{1}{\eta\cdot(k_1-k)} \Big(\gamma\cdot(k_1-k)\, \gamma\cdot k\, \gamma^+ +$$

$$\gamma^+ \gamma\cdot k\, \gamma\cdot(k_1-k) \Big) \Big] = 2k^+ \Big[-2\gamma\cdot k - \frac{1}{\eta\cdot(k_1-k)} \cdot$$

$$\cdot \Big(-2k^2 \gamma^+ + \gamma\cdot k_1\, \gamma\cdot k\, \gamma^+ + \gamma^+ \gamma\cdot k\, \gamma\cdot k_1 \Big) \Big]$$

we want to swap and move over here.

$$② = 2k^+ \Big[-2\gamma\cdot k - \frac{1}{\eta\cdot(k_1-k)} \Big(-2k^2\gamma^+ + 2k\cdot k_1\, \gamma^+ +$$

$$+ 2k^+ \gamma\cdot k_1 - 2k_1^+ \gamma\cdot k \Big) \Big]$$

Now we are interested in the regime where

$$|\underline{k}| \gg |\underline{k}_1|, \quad \underline{k}^2 \gg k_1^2, \quad \text{i.e. } |\underline{k}| \text{ is VERY LARGE.}$$

at the leading order in $|\underline{k}|$: $\quad k^- \approx -\dfrac{\underline{k}^2}{2k_1^+(1-z)}$

For large $|\underline{k}|$: ① $\simeq -2\gamma^+ \underline{k}^2 \frac{1}{1-z}$

as $k^2 = z k_1^2 - \frac{1}{1-z}(\underline{k} - z\underline{k}_1)^2 \to -\frac{\underline{k}^2}{1-z}$

$$② \simeq 2z\left[+\cancel{\not{k}}\gamma^+ \frac{\underline{k}^2}{\cancel{z}(1-z)} - \frac{1}{1-z}\left(\frac{2\underline{k}^2}{1-z}\gamma^+ - \overset{-2k^{-}k_1^+\gamma^+}{\cancel{\frac{\underline{k}^2}{1-z}\gamma^+}} + \right.\right.$$

$$\left.\left. + \underbrace{\cancel{\frac{\underline{k}^2}{1-z}\gamma^+}}_{\simeq -2k_1^+\gamma^+ k^-}\right)\right] = 2z\gamma^+ \frac{\underline{k}^2}{1-z}\left[1 - \frac{2}{1-z}\right] =$$

$$= -2\gamma^+ \underline{k}^2 \frac{z(1+z)}{(1-z)^2}$$

We assume transverse momentum ordering:

$$\boxed{Q^2 \gg k_\perp^2 \gg \underline{k}_1^2, k_1^2 \gg \Lambda_{QCD}^2}$$

$$\Rightarrow ① + ② = -2\gamma^+ \underline{k}^2 \frac{1+z^2}{(1-z)^2}$$

Plugging it all back we get

$$\Gamma^+ = -\frac{\alpha_s C_F}{4\pi^2}\delta_{\alpha\delta}\int \underbrace{\frac{d^2k}{\underline{k}^4}}_{(1/k^2)^2}(1-z)^2(-2)\gamma^+\underline{k}^2 \frac{1+z^2}{(1-z)^2}\frac{p^+/k_1^+}{1-z}$$

$\Rightarrow$ defining Bjorken (or Feynman) x for quark k_1

as $x_1 \equiv \frac{k_1^+}{p^+}$ we get

$$\boxed{\Gamma^+ = \gamma^+ \frac{1}{x_1}\delta_{\alpha\delta}\frac{\alpha_s C_F}{2\pi}\int \frac{d\underline{k}^2}{\underline{k}^2}\frac{1+z^2}{1-z}}$$

$$\Gamma^+ = \gamma^+ \frac{1}{x_1} \delta_{\alpha\beta} \frac{\alpha_s C_F}{2\pi} \int_{\underline{k}_1^2}^{Q^2} \frac{d\underline{k}^2}{\underline{k}^2} \frac{1+z^2}{1-z}$$ ~ putting the proper integration limits in

$$\Gamma^+ \sim \alpha_s \cdot \ln\left(Q^2/\underline{k}_1^2\right) \sim \alpha_s \ln Q^2/\Lambda^2$$

$\alpha_s << 1$ (perturbation theory, small coupling)

$\ln\left(Q^2/\Lambda^2\right) >> 1$ (DIS with large Q^2)

$\alpha \ln \frac{Q^2}{\Lambda^2} \sim 1$ our resummation parameter!

"<u>L</u>eading <u>L</u>ogarithmic <u>A</u>pproximation"

Remember: we neglected terms suppressed by $\frac{\underline{k}_1^2}{\underline{k}^2}, \frac{\underline{k}_1^4}{\underline{k}^4}, \ldots \Rightarrow$ they give

$$\int_{\underline{k}_1^2}^{Q^2} \frac{dk^2}{k^4} \underline{k}_1^2 \sim \left(\frac{1}{\underline{k}_1^2} - \frac{1}{Q^2}\right) \underline{k}_1^2 \sim 1 - \frac{\underline{k}_1^2}{Q^2}$$

no log; higher twist

Old (LO) Parton Model vertex (Mueller vertex) was $\gamma^+ \delta\left(x - \frac{k^+}{p^+}\right)$ ~ same γ_+ matrix as Γ_+

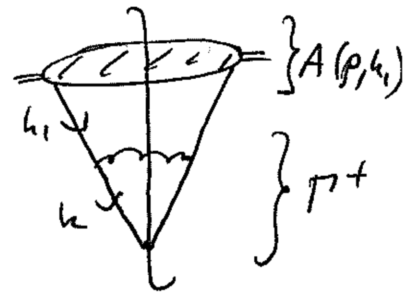
$$\Rightarrow \boxed{Q^2 \frac{\partial}{\partial Q^2} q^f(x, Q^2) = \frac{\alpha}{2\pi} \int_x^1 \frac{dx_1}{x_1} P_{qq}\left(\frac{x}{x_1}\right) q^f(x_1, Q^2)}$$

Start with the definition of quark PDF,

$$q^f(x,Q^2) = \frac{1}{2p^+}\int \frac{d^4k}{(2\pi)^4} A^f_{\alpha\beta}(p,k)\left(\gamma^+\right)_{\beta\alpha}\, \delta\left(x-\frac{k^+}{p^+}\right)$$

$\Downarrow$ correction is obtained by replacing $\gamma^+\delta(x-\frac{k^+}{p^+}) \to \Gamma^+$

$$\delta q^f(x,Q^2) = \frac{1}{2p^+}\int \frac{d^4k_1}{(2\pi)^4} A^f_{\alpha\beta}(p,k_1)\left(\Gamma^+\right)_{\beta\alpha}$$



$\Downarrow$ plug in Γ^+ we found:

$$\delta q^f(x,Q^2) = \frac{1}{2p^+}\int \frac{d^4k_1}{(2\pi)^4} A^f_{\alpha\beta}(p,k_1)\left(\gamma^+\right)_{\beta\alpha} \frac{p^+}{k_1^+}\cdot \frac{\alpha C_F}{2\pi}\int_{k_{\perp 1}^2}^{Q^2} \frac{dk_\perp^2}{k_\perp^2}\left(\frac{1+z^2}{1-z}\right)_+$$

Rewrite $\frac{p^+}{k_1^+} = \int_x^1 \frac{dx_1}{x_1}\,\delta\left(x_1-\frac{k_1^+}{p^+}\right)$ with x_1 a dummy variable.

(Note that $k_1^+ > k^+ \Rightarrow \frac{k_1^+}{p^+} > x \Rightarrow 1 > x > x_1$ is the right range of integration.)

$$\delta q^f(x,Q^2) = \int_x^1 \frac{dx_1}{x_1}\cdot \underbrace{\frac{1}{2p^+}\int \frac{d^4k_1}{(2\pi)^4} A^f_{\alpha\beta}(p,k_1)\left(\gamma^+\right)_{\beta\alpha}\,\delta\left(x_1-\frac{k_1^+}{p^+}\right)}_{q^f(x_1,\,k_\perp^2)\ \text{as}\ k_{\perp 1}^2 < k_\perp^2}\cdot$$

$$\cdot\,\frac{\alpha C_F}{2\pi}\int_{k_{\perp 1}^2}^{Q^2} \frac{dk_\perp^2}{k_\perp^2}\left(\frac{1+z^2}{1-z}\right)_+$$

$k_{\perp 1}^2 \to \Lambda^2$ ~ replace with LLA accuracy

$$\Rightarrow \delta q^f(x,Q^2) = q^f(x,Q^2) - q^f(x,\Lambda^2) = \frac{\alpha C_F}{2\pi}\int_x^1 \frac{dx_1}{x_1}\left(\frac{1+z^2}{1-z}\right)_+ \cdot$$

$$\cdot\int_{\Lambda^2}^{Q^2} \frac{dk_\perp^2}{k_\perp^2}\, q^f(x_1,k_\perp^2) \quad \text{with} \quad z = \frac{x}{x_1} = \frac{k^+}{k_1^+}$$

Differentiating both sides w.r.t. $\frac{\partial}{\partial \ln Q^2}$ we get

$$\boxed{\frac{\partial}{\partial \ln Q^2} q^f(x, Q^2) = \frac{\alpha C_F}{2\pi} \int_x^1 \frac{dx_1}{x_1} \left(\frac{1 + \left(\frac{x}{x_1}\right)^2}{1 - \frac{x}{x_1}} \right)_+ q^f(x_1, Q^2).}$$

Defining $P_{qq}(z) = C_F \left(\frac{1+z^2}{1-z} \right)_+$ we get

$$\boxed{\frac{\partial}{\partial \ln Q^2} q^f(x, Q^2) = \frac{\alpha_s}{2\pi} \int_x^1 \frac{dx_1}{x_1} P_{qq}\left(\frac{x_1}{x}\right) q^f(x_1, Q^2)}.$$

or, equivalently,

$$\boxed{\frac{\partial}{\partial \ln Q^2} q^f(x, Q^2) = \frac{\alpha_s}{2\pi} \int_x^1 \frac{dz}{z} P_{qq}(z) q^f\left(\frac{x}{z}, Q^2\right)}$$

$$P_{qq}(z) = C_F \left[\frac{1+z^2}{(1-z)_+} + \frac{3}{2} \delta(1-z) \right]$$ ~ another form

where $x = \frac{k^+}{p^+}$, $x_1 = \frac{k_1^+}{p^+} \Rightarrow z = \frac{k^+}{k_1^+} = \frac{x}{x_1}$.

as $z < 1 \Rightarrow x_1 > x$ in the integral.

Including the virtual terms (B and C)

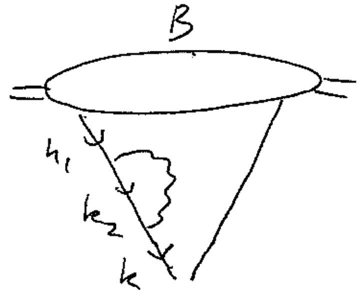
gives

$$P_{qq}(z) = C_F \left(\frac{1+z^2}{1-z}\right)_+ \quad \sim \underline{\text{splitting function}}$$

where

$$\int_0^1 dz \, [h(z)]_+ \, f(z) = \int_0^1 dz \, h(z) \, [f(z) - f(1)]$$

Easy to understand:



$\propto \delta(k - k_1)$
$\Downarrow$
$x = x_1$

$\Rightarrow q_f(x_1, Q^2) = q_f(x, Q^2)$
as $x = x_1$

"real" part, diagram A

$$Q^2 \frac{\partial}{\partial Q^2} q^f(x, Q^2) = \frac{\alpha C_F}{2\pi} \left[\int_x^1 \frac{dz}{z} \frac{1+z^2}{1-z} \cdot q^f\!\left(\frac{x}{z}, Q^2\right) - \int_0^1 dz \frac{1+z^2}{1-z} q^f(x, Q^2) \right]$$

Virtual corrections, graphs B & C

$z \to 1$ divergence is cancelled between the real (A) and virtual (B+C) terms.

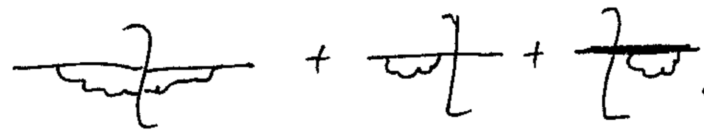
bare quark state $|\psi_0\rangle = $ —— $\Rightarrow \langle\psi_0|\psi_0\rangle = 1$

(normalization)

dressed quark state $|\psi\rangle =$ 

$|\psi_0\rangle$ $|\psi_1\rangle$ $V|\psi_0\rangle$

normalization:

$\langle\psi|\psi\rangle = 1 = \langle\psi_0|\psi_0\rangle +$ 

$$= 1 + \langle\psi_1|\psi_1\rangle + 2V\langle\psi_0|\psi_0\rangle = 1 + \langle\psi_1|\psi_1\rangle + 2V$$

$$\Rightarrow V = -\frac{1}{2}\langle\psi_1|\psi_1\rangle$$

$\Rightarrow$ graphs $B, C = -\frac{1}{2}A \Rightarrow B + C = -A$.

~ simply imposed probability conservation!

$$\frac{\partial}{\partial \ln Q^2} q^f(x,Q^2) = \frac{\alpha C_F}{2\pi} \int_x^1 \frac{dx_1}{x_1} \left(\frac{1+\left(\frac{x}{x_1}\right)^2}{1-\frac{x}{x_1}} \right)_+ q^f(x_1,Q^2) =$$

$$= \left| \begin{array}{l} z = \frac{x}{x_1} \\ \frac{dx_1}{x_1} = -\frac{dz}{z} \end{array} \right. = \frac{\alpha C_F}{2\pi} \int_x^1 \frac{dz}{z} \left(\frac{1+z^2}{1-z} \right)_+ q^f\left(\frac{x}{z}, Q^2\right)$$

$$= \frac{\alpha C_F}{2\pi} \int_0^1 dz \underbrace{\left(\frac{1+z^2}{1-z} \right)_+}_{h(z)} \underbrace{\frac{1}{z} q^f\left(\frac{x}{z}, Q^2\right) \theta(z-x)}_{f(z)}$$

$$= \left| \begin{array}{l} \text{using} \\ \int_0^1 dz \left[h(z)\right]_+ f(z) = \int_0^1 dz\, h(z) \left[f(z) - f(1)\right] \end{array} \right.$$

$$= \frac{\alpha C_F}{2\pi} \int_0^1 dz \frac{1+z^2}{1-z} \left[\frac{1}{z} q^f\left(\frac{x}{z}\right) \theta(z-x) - q^f(x) \underset{1}{\underset{\shortparallel}{\theta(1-x)}} \right]$$

$$= \frac{\alpha_s C_F}{2\pi} \left[\int_x^1 \frac{dz}{z} \frac{1+z^2}{1-z} q^f\left(\frac{x}{z}, Q^2\right) - \int_0^1 dz \frac{1+z^2}{1-z} q^f(x) \right]$$