Axial Anomaly

Consider massless QED as an example:

$$\mathcal{L} = \bar\psi i \gamma \cdot \partial \psi - e \bar\psi \gamma^\mu A_\mu \psi - \frac{1}{4} F_{\mu\nu} F^{\mu\nu}$$

$\psi \sim$ electron field, $A_\mu \sim$ photon field.

$\mathcal{L}$ is invariant under the following *global* symmetries:

(i) $\psi \to e^{i\alpha} \psi \Rightarrow \bar\psi \to e^{-i\alpha} \bar\psi \Rightarrow \mathcal{L}$ is invariant under $U(1)$ global symmetry.

The corresponding current is $j_\mu = \bar\psi \gamma^\mu \psi$.

It is conserved: $\partial_\mu j^\mu = 0$

as $\gamma_5^+ = \gamma_5$

(ii) $\psi \to e^{i\gamma_5 \alpha} \psi \Rightarrow \bar\psi = \psi^+ \gamma^0 \to \psi^+ e^{-i\gamma_5 \alpha} \gamma^0$

$\{\gamma_5, \gamma^0\} = 0$

$= \psi^+ \gamma^0 e^{i\gamma_5 \alpha} = \bar\psi e^{i\alpha\gamma_5} \Rightarrow$

$$\bar\psi i \gamma^\mu D_\mu \psi \to \bar\psi e^{i\alpha\gamma_5} i\gamma^\mu D_\mu e^{i\alpha\gamma_5} \psi =$$

$$= \bar\psi e^{i\alpha\gamma_5} e^{-i\alpha\gamma_5} i\gamma^\mu D_\mu \psi = \bar\psi i\gamma^\mu D_\mu \psi$$

as $\{\gamma_5, \gamma^\mu\} = 0$

$\Rightarrow$ corresponding conserved current is

(massless fermions)

$j_\mu^5 = \bar\psi \gamma^\mu \gamma_5 \psi$: $\partial_\mu j^{5\mu} = 0$

$\Rightarrow$ seems like massless QED Lagrangian is invariant under the axial symmetry $U_A(1)$

$\Rightarrow \mathcal{L}_{QED}$ is $U(1) \otimes U_A(1)$ invariant.

However, this is <u>not true</u> when quantum corrections are included. $\Rightarrow$ we will see that $\partial_\mu j^{5\mu} \neq 0$ if quantum corrections are counted.

Consider $\partial_\mu j^{5\mu} = \partial_\mu (\bar{\psi} \gamma^\mu \gamma_5 \psi) \Rightarrow$ in momentum space $\partial_\mu \to -i k_\mu$, the vertex has $\gamma^\mu \gamma_5$.

Consider 3-point correlator:

$$T_{\mu\nu\rho}(k_1, k_2) = i \int d^4x_1\, d^4x_2\, e^{ik_1 \cdot x_1 + ik_2 \cdot x_2} \langle 0 | T\big(j_\mu(x_1)\, j_\nu(x_2) \cdot j^5_\rho(0)\big) | 0 \rangle$$

~~Writing $T_{\mu\nu\rho}(k_1,k_2) = i\int d^4x_1\, d^4x_2\, e^{i(x_1-y)\cdot k_1 + ik_2\cdot(x_2-y)}$~~

~~$\langle 0|T[j_\mu(x_1)\, j_\nu(x_2)\, j^5_\rho(y)]|0\rangle$ $\Rightarrow$ we expect~~

~~$(k_1+k_2)^\rho T_{\mu\nu\rho}(k_1,k_2) = i\int d^4x_1\, d^4x_2\, i\partial^\rho_y \left[e^{ik_1\cdot(x_1-y)+ik_2\cdot(x_2-y)}\right]$~~

~~$\langle 0|T[j_\mu(x)\, j_\nu(x)\, j^5_\rho(y)]|0\rangle$ = (parts)~~

One can show that $\partial_\mu j^{5\mu} = 0$ would lead to

$$(k_1 + k_2)^\rho\, T_{\mu\nu\rho}(k_1, k_2) = 0.$$

$$\int_0^\infty dx = \Big| x \to x + a = y = \int_a^\infty dy = -\int_0^a dy + \int_0^\infty dy = -a + \int_0^\infty dy$$

$\Rightarrow a = 0$ for $\forall$ a $\Rightarrow$ all #'s are zero!

Weyl basis: $\psi_D = \begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix}$, $\gamma_5 = \begin{pmatrix} -\mathbb{1}_{2\times 2} & 0 \\ 0 & \mathbb{1}_{2\times 2} \end{pmatrix}$

$$\Rightarrow \mathcal{L} = \bar{\psi} i \not{D} \psi - \frac{1}{4} F_{\mu\nu} F^{\mu\nu}$$

$U(1)$: $\psi(x) \to e^{i\alpha} \psi(x) \Rightarrow \begin{cases} \psi_L \to e^{i\alpha} \psi_L \\ \psi_R \to e^{i\alpha} \psi_R \end{cases}$

$U_A(1)$: $\psi(x) \to \underbrace{e^{i\alpha' \gamma_5}}_{= \begin{pmatrix} e^{-i\alpha'} & 0 \\ 0 & e^{i\alpha'} \end{pmatrix}} \psi(x) \Rightarrow \begin{cases} \psi_L \to e^{-i\alpha'} \psi_L \\ \psi_R \to e^{i\alpha'} \psi_R \end{cases}$

$$\mathcal{L} = \bar{\psi}_L i \not{D} \psi_L + \bar{\psi}_R i \not{D} \psi_R$$

$$\Rightarrow \psi_L \to e^{i\alpha_1} \psi_L \qquad \psi_R \to e^{i\alpha_2} \psi_R$$

$$U(1)_L \otimes U(1)_R$$

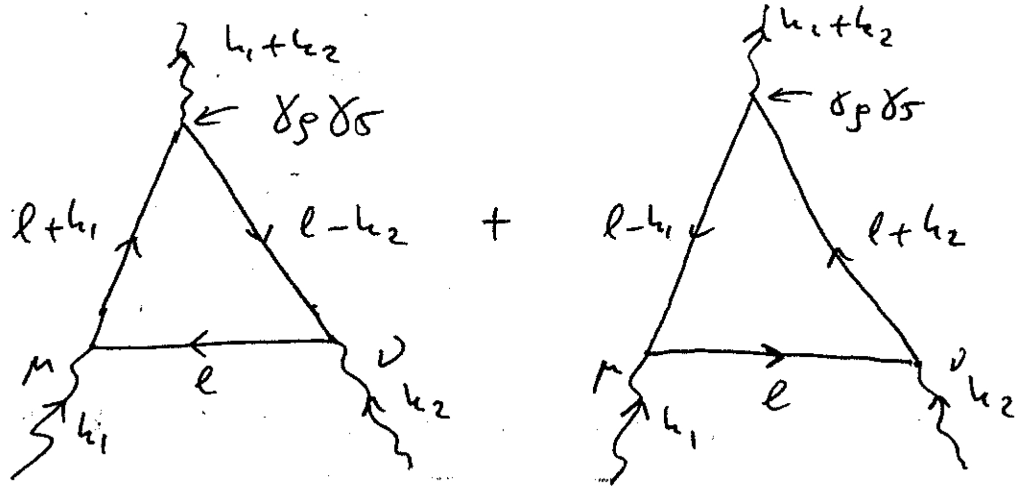
different bases

or $U(1) \otimes U_A(1)$

Check this statement:

$T_{\mu\nu\rho}(k_1,k_2) =$

arrow indicates both momentum & fermion #.



graph A graph B

(Can write $T_{\mu\nu\rho}(k_1,k_2) \underset{\substack{x_1\to 0\\ 0\to x_1}}{=} i\int d^4x_1\, d^4x_2\, e^{-ik_1\cdot x_1 + ik_2\cdot(x_2-x_1)}$

$\cdot\langle 0|T[j_\mu(0)\, j_\nu(x_2)\, j^5_\rho(x_1)]|0\rangle \Rightarrow (k_1+k_2)^\rho\, T_{\mu\nu\rho} =$

$= i\int d^4x_1\, d^4x_2\; i\,\partial^\rho_{x_1}\left(e^{ik_2\cdot x_2 - i(k_1+k_2)\cdot x_1}\right)\langle 0|T[j_\mu(0)j_\nu(x_2)j^5_\rho(x_1)]|0\rangle$

$\Rightarrow$ (parts) $= \int d^4x_1\, d^4x_2\, e^{ik_2\cdot x_2 - i(k_1+k_2)\cdot x_1}\langle 0|T[j_\mu(0)j_\nu(x_2)\cdot$
+more work (equal-time commut. relations, etc) ~ see attached pages

$\cdot\,\partial^\rho j^5_\rho(x_1)]|0\rangle = 0$ if $\partial^\rho j^5_\rho = 0$.)

fermion loop, $m=0$

$$-iT_{\mu\nu\rho} = -(-ie)^2\int\frac{d^4l}{(2\pi)^4}\,\mathrm{Tr}\left[\gamma_\rho\gamma_5\frac{i}{\not l+\not k_1}\gamma_\mu\frac{i}{\not l}\gamma_\nu\frac{i}{\not l-\not k_2}\right]$$

$$-(-ie)^2\int\frac{d^4l}{(2\pi)^4}\,\mathrm{Tr}\left[\gamma_\rho\gamma_5\frac{i}{\not l+\not k_2}\gamma_\nu\frac{i}{\not l}\gamma_\mu\frac{i}{\not l-\not k_1}\right] =$$

$$= -ie^2\int\frac{d^4l}{(2\pi)^4}\,\frac{\mathrm{Tr}[\gamma_\rho\gamma_5(\not l+\not k_1)\gamma_\mu\not l\gamma_\nu(\not l-\not k_2)]}{(l^2+i\varepsilon)((l+k_1)^2+i\varepsilon)((l-k_2)^2+i\varepsilon)} -$$

$$-ie^2\int\frac{d^4\ell}{(2\pi)^4}\,\frac{\mathrm{Tr}[\gamma_\rho\gamma_5(\not\ell+\not k_2)\gamma_\nu\not\ell\gamma_\mu(\not\ell-\not k_1)]}{(\ell^2+i\varepsilon)((\ell+k_2)^2+i\varepsilon)((\ell-k_1)^2+i\varepsilon)}$$ (118.)

$$\Rightarrow (k_1+k_2)^\rho\, T_{\mu\nu\rho} = e^2\int\frac{d^4\ell}{(2\pi)^4}\left\{\underbrace{\frac{\mathrm{Tr}[(\not k_1+\not k_2)\gamma_5(\not\ell+\not k_1)\gamma_\mu\not\ell\gamma_\nu(\not\ell-\not k_2)]}{(\ell^2+i\varepsilon)((\ell+k_1)^2+i\varepsilon)((\ell-k_2)^2+i\varepsilon)}}_{=A}\right.$$

$$\left.+\underbrace{\frac{\mathrm{Tr}[(\not k_1+\not k_2)\gamma_5(\not\ell+\not k_2)\gamma_\nu\not\ell\gamma_\mu(\not\ell-\not k_1)]}{(\ell^2+i\varepsilon)((\ell+k_2)^2+i\varepsilon)((\ell-k_1)^2+i\varepsilon)}}_{=B}\right\}$$

Numerator of A $= \mathrm{Tr}\Big[(\not k_1+\not\ell-(\not\ell-\not k_2))\gamma_5(\not\ell+\not k_1)\gamma_\mu\not\ell$ $\gamma_\nu(\not\ell-\not k_2)\Big] = -(\ell+k_1)^2\,\mathrm{Tr}[\gamma_5\gamma_\mu\not\ell\gamma_\nu(\not\ell-\not k_2)]-$ $-(\ell-k_2)^2\,\mathrm{Tr}[\gamma_5(\not\ell+\not k_1)\gamma_\mu\not\ell\gamma_\nu]$

Numerator of B $= \mathrm{Tr}\Big[((\not k_2+\not\ell)-(\not\ell-\not k_1))\gamma_5(\not\ell+\not k_2)\cdot$ $\cdot\gamma_\nu\not\ell\gamma_\mu(\not\ell-\not k_1)\Big] = -(\ell+k_2)^2\,\mathrm{Tr}[\gamma_5\gamma_\nu\not\ell\gamma_\mu(\not\ell-\not k_1)]$ $-(\ell-k_1)^2\,\mathrm{Tr}[\gamma_5(\not\ell+\not k_2)\gamma_\nu\not\ell\gamma_\mu] \Rightarrow$

$$(k_1+k_2)^\rho\, T_{\mu\nu\rho} = -e^2\int\frac{d^4\ell}{(2\pi)^4}\,\frac{1}{\ell^2+i\varepsilon}\left\{\frac{\mathrm{Tr}[\gamma_5\gamma_\mu\not\ell\gamma_\nu(\not\ell-\not k_2)]}{(\ell-k_2)^2+i\varepsilon}^{(1)}+\right.$$

$$+\frac{\mathrm{Tr}[\gamma_5(\not\ell+\not k_1)\gamma_\mu\not\ell\gamma_\nu]}{(\ell+k_1)^2+i\varepsilon}^{(2)}+\frac{\mathrm{Tr}[\gamma_5\gamma_\nu\not\ell\gamma_\mu(\not\ell-\not k_1)]}{(\ell-k_1)^2+i\varepsilon}^{(3)}+$$

$$\left.+\frac{\mathrm{Tr}[\gamma_5(\not\ell+\not k_2)\gamma_\nu\not\ell\gamma_\mu]}{(\ell+k_2)^2+i\varepsilon}^{(4)}\right\}$$

$$T_{\mu\nu\rho}(k_1,k_2) = i\int d^4x_1\, d^4x_2\, e^{-ik_1\cdot x_1 + ik_2\cdot(x_2-x_1)} \cdot$$

$$\cdot \langle 0|T\{j_\mu(0)\, j_\nu(x_2)\, j^5_\rho(x_1)\}|0\rangle \quad \Rightarrow$$

$$(k_1+k_2)^\rho T_{\mu\nu\rho} = i\int d^4x_1\, d^4x_2 \;\; i\,\partial^\rho_{x_1}\left(e^{ik_2\cdot x_2 - i(k_1+k_2)\cdot x_1}\right) \cdot$$

$$\cdot\langle 0|T\{j_\mu(0)\, j_\nu(x_2)\, j^5_\rho(x_1)\}|0\rangle = \Big|\text{parts}$$

$$= \int d^4x_1\, d^4x_2\; e^{ik_2\cdot x_2 - i(k_1+k_2)\cdot x_1}\langle 0|\,\partial^\rho_{x_1}\cdot T\{j_\mu(0)\cdot$$

$$\cdot j_\nu(x_2)\, j^5_\rho(x_1)\}|0\rangle.$$

Now, $T\{j_\mu(0)\, j_\nu(x_2)\, j^5_\rho(x_1)\} = \theta(-x_2^0)\,\theta(x_2^0-x_1^0)\cdot$

$$\cdot j_\mu(0)\, j_\nu(x_2)\, j^5_\rho(x_1) + \theta(-x_1^0)\,\theta(x_1^0-x_2^0)\, j_\mu(0)\, j^5_\rho(x_1)\, j_\nu(x_2)$$

$$+\ \theta(x_2^0)\theta(-x_1^0)\, j_\nu(x_2)\, j_\mu(0)\, j^5_\rho(x_1) + \theta(x_2^0-x_1^0)\theta(x_1^0)\, j_\nu(x_2)\, j^5_\rho(x_1)\, j_\mu(0)$$

$$+\ \theta(x_1^0)\,\theta(-x_2^0)\, j^5_\rho(x_1)\, j_\mu(0)\, j_\nu(x_2) + \theta(x_1^0-x_2^0)\theta(x_2^0)\, j^5_\rho(x_1)\, j_\nu(x_2)\, j_\mu(0)$$

$$\Rightarrow \partial^\rho_{x_1} T\{j_\mu(0)\, j_\nu(x_2)\, j^5_\rho(x_1)\} = T\{j_\mu(0)\, j_\nu(x_2)\, \overbrace{\partial^\rho_{x_1} j^5_\rho(x_1)}^{=0 \text{ if conserved}}\} +$$

$$+\ \delta^{\rho 0}\Big[-\theta(-t_2)\,\delta(t_1-t_2)\, j_\mu(0)\, j_\nu(x_2)\, j^5_\rho(x_1) - \delta(t_1)\,\theta(-t_2)\, j_\mu(0)\cdot$$

$$\cdot j^5_\rho(x_1)\, j_\nu(x_2) + \theta(-t_2)\,\delta(t_1-t_2)\, j_\mu(0)\, j^5_\rho(x_1)\, j_\nu(x_2)$$

(117'')

$$- \theta(t_2)\,\delta(t_1)\, j_\nu(x_2)\, \dot{j}_\mu(0)\, j_\rho^5(x_1) - \delta(t_1-t_2)\,\theta(t_2)\, j_\nu(x_2) j_\rho^5(x_1) j_\mu(0)$$

$$+ \theta(t_2)\,\delta(t_1)\, j_\nu(x_2)\, j_\rho^5(x_1)\, j_\mu(0) + \delta(t_1)\,\theta(-t_2)\, j_\rho^5(x_1)\, j_\mu(0)\, j_\nu(x_2)$$

$$+ \delta(t_1-t_2)\,\theta(t_2)\, j_\rho^5(x_1)\, j_\nu(x_2) j_\mu(0) \Big] = \delta^{\rho 0} \cdot \Big[-\theta(-t_2)\,\delta(t_1-t_2)$$

$$\cdot j_\mu(0)\,[\,j_\nu(x_2), j_\rho^5(x_1)] - \theta(t_2)\,\delta(t_1-t_2)\,[\,j_\nu(x_2), j_\rho^5(x_1)]\, j_\mu(0)$$

$$- \theta(-t_2)\,\delta(t_1)\,[\,j_\mu(0), j_\rho^5(x_1)]\, j_\nu(x_2) - \theta(t_2)\,\delta(t_1) \cdot j_\nu(x_2)$$

$$\cdot [\,j_\mu(0), j_\rho^5(x_1)] \Big] = 0$$

The expression is zero because all the equal-time correlation relations are zero for $\rho = 0$. For instance,

$$\delta^{\rho 0}\delta(t_1-t_2)\,[\,j_\nu(x_2), j_\rho^5(x_1)] = \delta^4(x_1-x_2)\,\psi^+ [\gamma^0\gamma^\nu, \gamma^0\gamma^0\gamma^5]\psi$$

last semester, problem 2, HW5, part a

$$= \delta^4(x_1-x_2)\,\psi^+ [\gamma^0\gamma^\nu, \gamma^5]\psi = 0 \quad \text{as } [\gamma^0\gamma^\nu, \gamma^5] = 0.$$

(119')

Shifts in ill-defined integrals:

$$\int_0^\infty dx = \Big|\, x \to x + a = y \;=\; \int_a^\infty dy = -\int_0^a dy + \int_0^\infty dy = -a + \int_0^\infty dy$$

$\Rightarrow a = 0$ for $\forall$ real a $\Rightarrow$ all #'s are zero.