Last time / **Axial Anomaly** (cont'd)

massless QED : $\mathcal{L} = \bar{\psi}\, i \not{\partial} \psi - \frac{1}{4} F_{\mu\nu} F^{\mu\nu}$

$U(1)$: $\psi(x) \to e^{i\alpha}\psi(x)$ global

$U(1)_A$: $\psi(x) \to e^{i\gamma_5 \alpha}\psi(x)$ global

$j_\mu = \bar{\psi}\gamma_\mu \psi$

$j_\mu{}^5 = \bar{\psi}\gamma_\mu \gamma^5 \psi$

$\partial_\mu j^\mu = 0$

$\partial_\mu j_5{}^\mu = 0$ $(m=0)$

current conservation is true classically.

Is it true when quantum loops are included?

$$T_{\mu\nu\rho}(k_1,k_2) \equiv i \int d^4x_1\, d^4x_2\, e^{ik_1\cdot x_1 + ik_2 \cdot x_2} \cdot \langle 0 | T\left[j_\mu(x_1)\, j_\nu(x_2)\, j_\rho^5(0)\right] |0\rangle$$

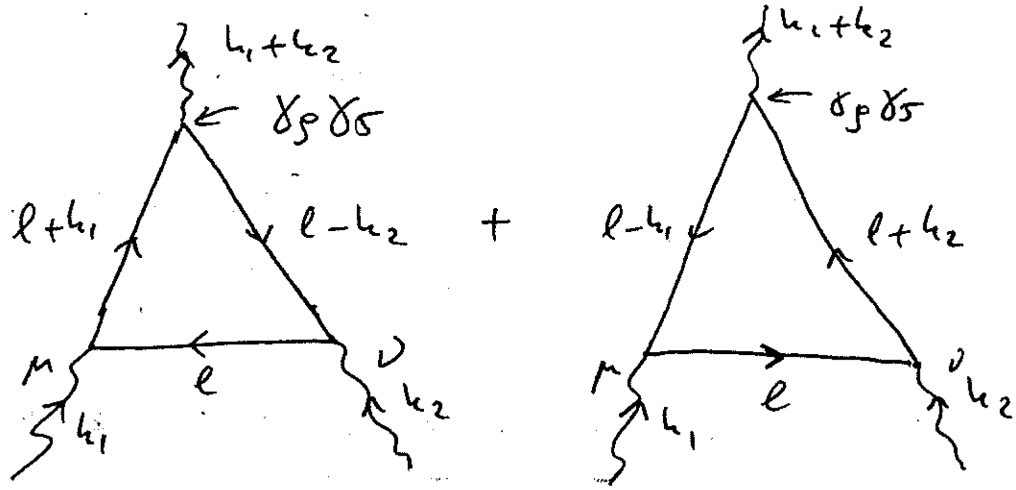
if $\partial_\mu j_5{}^\mu = 0 \Rightarrow (k_1+k_2)^\rho\, T_{\mu\nu\rho}(k_1,k_2) = 0.$

(can prove)

Check this statement:

$T_{\mu\nu\rho}(k_1, k_2) =$

arrow indicates both momentum & fermion #.



graph A   graph B

((Can write $T_{\mu\nu\rho}(k_1, k_2) \underset{\substack{x_1 \to 0 \\ 0 \to x_1}}{=} i \int d^4x_1\, d^4x_2\, e^{-ik_1\cdot x_1 + ik_2\cdot(x_2-x_1)}$

$\cdot \langle 0 | T[j_\mu(0)\, j_\nu(x_2)\, j_\rho^5(x_1)] | 0\rangle \Rightarrow (k_1+k_2)^\rho\, T_{\mu\nu\rho} =$

$= i \int d^4x_1\, d^4x_2\; i\,\partial^\rho_{x_1}\left(e^{ik_2\cdot x_2 - i(k_1+k_2)\cdot x_1}\right) \langle 0 | T\{j_\mu(0)\, j_\nu(x_2)\, j_\rho^5(x_1)\} |0\rangle$

$\Rightarrow$ (parts) $= \int d^4x_1\, d^4x_2\, e^{ik_2\cdot x_2 - i(k_1+k_2)\cdot x_1} \langle 0 | T[j_\mu(0)\, j_\nu(x_2)\,\cdot$

+ more work (equal-time commut. relations, etc) ~ see attached pages

$\cdot\, \partial^\rho j_\rho^5(x_1)] | 0\rangle = 0$ if $\partial^\rho j_\rho^5 = 0$.)

fermion loop, $m=0$

$$-i\,T_{\mu\nu\rho} = -(-ie)^2 \int \frac{d^4l}{(2\pi)^4}\, \mathrm{Tr}\left[\gamma_\rho\gamma_5 \frac{i}{\not l + \not k_1} \gamma_\mu \frac{i}{\not l} \gamma_\nu \frac{i}{\not l - \not k_2}\right]$$

We put $j_\mu = -ie\bar\psi\gamma_\mu\psi$ to have a real QED vertex.

$$-(-ie)^2 \int \frac{d^4l}{(2\pi)^4}\, \mathrm{Tr}\left[\gamma_\rho\gamma_5 \frac{i}{\not l + \not k_2} \gamma_\nu \frac{i}{\not l} \gamma_\mu \frac{i}{\not l - \not k_1}\right] =$$

$$= -ie^2 \int \frac{d^4l}{(2\pi)^4} \frac{\mathrm{Tr}[\gamma_\rho\gamma_5(\not l + \not k_1)\gamma_\mu \not l \gamma_\nu(\not l - \not k_2)]}{(l^2+i\varepsilon)((l+k_1)^2+i\varepsilon)((l-k_2)^2+i\varepsilon)} -$$

$$-ie^2\int\frac{d^4\ell}{(2\pi)^4}\,\frac{Tr[\gamma_\rho\gamma_5(\not\ell+\not k_2)\gamma_\nu\not\ell\gamma_\mu(\not\ell-\not k_1)]}{(\ell^2+i\varepsilon)((\ell+k_2)^2+i\varepsilon)((\ell-k_1)^2+i\varepsilon)}$$

$$\Rightarrow (k_1+k_2)^\rho\, T_{\mu\nu\rho} = e^2\int\frac{d^4\ell}{(2\pi)^4}\left\{\underbrace{\frac{Tr[(\not k_1+\not k_2)\gamma_5(\not\ell+\not k_1)\gamma_\mu\not\ell\gamma_\nu(\not\ell-\not k_2)]}{(\ell^2+i\varepsilon)((\ell+k_1)^2+i\varepsilon)((\ell-k_2)^2+i\varepsilon)}}_{\equiv A}\right.$$

$$\left.+\underbrace{\frac{Tr[(\not k_1+\not k_2)\gamma_5(\not\ell+\not k_2)\gamma_\nu\not\ell\gamma_\mu(\not\ell-\not k_1)]}{(\ell^2+i\varepsilon)((\ell+k_2)^2+i\varepsilon)((\ell-k_1)^2+i\varepsilon)}}_{\equiv B}\right\}$$

Numerator of $A = Tr\Big[\big(\not k_1+\not\ell-(\not\ell-\not k_2)\big)\gamma_5(\not\ell+\not k_1)\gamma_\mu\not\ell$

$\gamma_\nu(\not\ell-\not k_2)\Big] = -(\ell+k_1)^2\, Tr[\gamma_5\gamma_\mu\not\ell\gamma_\nu(\not\ell-\not k_2)]-$

$-(\ell-k_2)^2\, Tr[\gamma_5(\not\ell+\not k_1)\gamma_\mu\not\ell\gamma_\nu]$

Numerator of $B = Tr\Big[\big((\not k_2+\not\ell)-(\not\ell-\not k_1)\big)\gamma_5(\not\ell+\not k_2)\cdot$

$\cdot\gamma_\nu\not\ell\gamma_\mu(\not\ell-\not k_1)\Big] = -(\ell+k_2)^2\, Tr[\gamma_5\gamma_\nu\not\ell\gamma_\mu(\not\ell-\not k_1)]$

$-(\ell-k_1)^2\, Tr[\gamma_5(\not\ell+\not k_2)\gamma_\nu\not\ell\gamma_\mu] \Rightarrow$

$$(k_1+k_2)^\rho\, T_{\mu\nu\rho} = -e^2\int\frac{d^4\ell}{(2\pi)^4}\,\frac{1}{\ell^2+i\varepsilon}\left\{\frac{Tr[\gamma_5\gamma_\mu\not\ell\gamma_\nu(\not\ell-\not k_2)]}{(\ell-k_2)^2+i\varepsilon}^{(1)}+\right.$$

$$+\frac{Tr[\gamma_5(\not\ell+\not k_1)\gamma_\mu\not\ell\gamma_\nu]}{(\ell+k_1)^2+i\varepsilon}^{(2)}+\frac{Tr[\gamma_5\gamma_\nu\not\ell\gamma_\mu(\not\ell-\not k_1)]}{(\ell-k_1)^2+i\varepsilon}^{(3)}+$$

$$\left.+\frac{Tr[\gamma_5(\not\ell+\not k_2)\gamma_\nu\not\ell\gamma_\mu]}{(\ell+k_2)^2+i\varepsilon}^{(4)}\right\}$$

Now, if in 4 we shift $l \to l-k_2 \Rightarrow$ it would cancel 1 as $1 + 4 \propto \{\gamma_5, \gamma_\mu\} = 0$.

In 3 shift $l \to l+k_1 \Rightarrow$ cancel 2.

$\Rightarrow$ seems to get $(k_1+k_2)^\rho T_{\mu\nu\rho} = 0$ in expectation with $\partial^\rho j^5_\rho = 0 \ldots$

<u>Problem</u> at large l all integrals are quadratically divergent!

We get $1 \sim 2 \sim 3 \sim 4 \sim \int d^4 l \, \frac{1}{l^2} \sim \int dl \cdot l \sim \infty^2$.

$\Rightarrow$ can't shift variables in divergent integrals!

$$\int_0^\infty dl \cdot l \xrightarrow[\substack{\text{shift} \\ l \to l+a}]{} \int_{-a}^\infty dl \cdot (l+a) = \int_0^\infty dl \cdot (l+a) + \int_{-a}^0 dl \cdot (l+a)$$

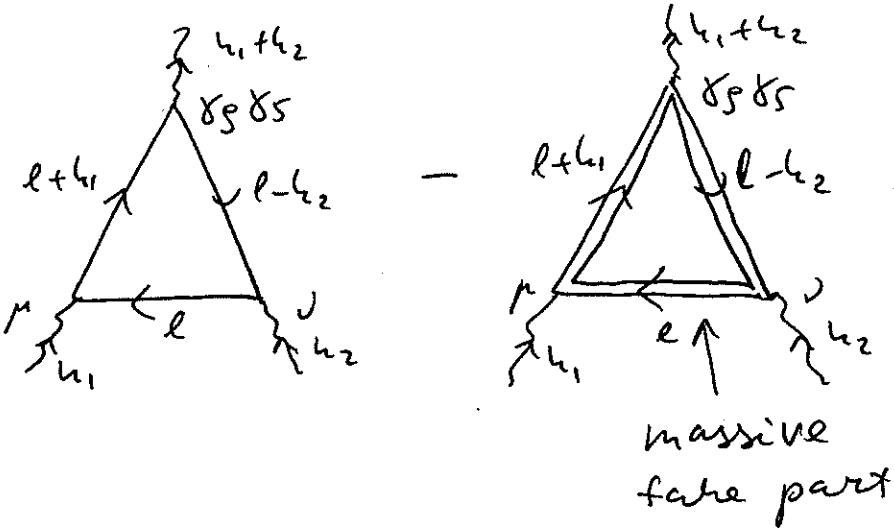
$$= \int_0^\infty dl \cdot l + a \cdot \int_0^\infty dl + \left(\frac{l^2}{2} + a l\right)\Bigg|_{-a}^0 = \underbrace{\int_0^\infty dl \cdot l}_{\text{old integral}} + a \underbrace{\int_0^\infty dl}_{\infty} + \frac{a^2}{2}$$

$\Rightarrow$ did not survive the shift, got corrections?

$\Rightarrow$ ill-defined procedure $\Rightarrow$ need to make integrals finite, need to regulate them!

We'll use Pauli-Villars regularization: introduce (subtract) a new particle with mass m, which is then taken to ∞ to eliminate the particle.

$T_{\mu\nu\rho} =$



$+ \underbrace{\begin{pmatrix} k_1 \leftrightarrow k_2 \\ \mu \leftrightarrow \nu \end{pmatrix}}_{\text{graph B}}$

$$T_{\mu\nu\rho} = -e^2 \int \frac{d^4\ell}{(2\pi)^4} \left\{ \frac{\mathrm{Tr}\left[\gamma_\rho\gamma_5 (\not\ell + \not k_1)\gamma_\mu \not\ell \gamma_\nu (\not\ell - \not k_2)\right]}{(\ell^2+i\varepsilon)((\ell+k_1)^2+i\varepsilon)((\ell-k_2)^2+i\varepsilon)} \right.$$

$$\left. - \frac{\mathrm{Tr}\left[\gamma_\rho\gamma_5(\not\ell+\not k_1+m)\gamma_\mu(\not\ell+m)\gamma_\nu(\not\ell-\not k_2+m)\right]}{(\ell^2-m^2+i\varepsilon)((\ell+k_1)^2-m^2+i\varepsilon)((\ell-k_2)^2-m^2+i\varepsilon)} \right\} + \begin{pmatrix} k_1\leftrightarrow k_2 \\ \mu\leftrightarrow\nu \end{pmatrix}$$

The second Tr has only even powers of m in its expansion (m^0 & m^2). (Tr of odd # of γ's is zero.) Write:

$$(k_1+k_2)^\rho\, T_{\mu\nu\rho} = -e^2 \int \frac{d^4\ell}{(2\pi)^4} \left\{ \mathrm{Tr}\left[(\not k_1+\not k_2)\gamma_5(\not\ell+\not k_1)\gamma_\mu\not\ell\gamma_\nu \cdot\right.\right.$$

$$\left. \cdot(\not\ell-\not k_2)\right]\left[\frac{1}{\ell^2(\ell+k_1)^2(\ell-k_2)^2} - \frac{1}{(\ell^2-m^2)[(\ell+k_1)^2-m^2][(\ell-k_2)^2-m^2]}\right] -$$

$$\left. - \frac{m^2\text{-term in 2nd trace } (O(\ell))}{[\ell^2-m^2][(\ell+k_1)^2-m^2][(\ell-k_2)^2-m^2]} \right\} + \begin{pmatrix} k_1\leftrightarrow k_2 \\ \mu\leftrightarrow\nu \end{pmatrix}$$

Now the integral is convergent & shifts are allowed!

$\Rightarrow$ the $m=0$ (first) term in $[\ldots]$ vanishes like before. (121)

for the term in $[\ldots]$ containing m^2 write:

$$\mathrm{Tr}\left[\left(\not{k}_1+\not{l}-(\not{l}-\not{k}_2)\right)\gamma_5(\not{l}+\not{k}_1)\gamma_\mu\not{l}\gamma_\nu(\not{l}-\not{k}_2)\right] =$$

$$= -(l+k_1)^2\,\mathrm{Tr}\left[\gamma_5\gamma_\mu\not{l}\gamma_\nu(\not{l}-\not{k}_2)\right]-(l-k_2)^2\,\mathrm{Tr}\left[\gamma_5(\not{l}+\not{k}_1)\cdot\gamma_\mu\not{l}\gamma_\nu\right] = -\left[(l+k_1)^2-m^2\right]\,\mathrm{Tr}\left[\gamma_5\gamma_\mu\not{l}\gamma_\nu(\not{l}-\not{k}_2)\right]$$

$$-\left[(l-k_2)^2-m^2\right]\mathrm{Tr}\left[\gamma_5(\not{l}+\not{k}_1)\gamma_\mu\not{l}\gamma_\nu\right]-m^2\left(\mathrm{Tr}\left[\gamma_5\gamma_\mu\not{l}\gamma_\nu\cdot(\not{l}-\not{k}_2)\right]+\mathrm{Tr}\left[\gamma_5(\not{l}+\not{k}_1)\gamma_\mu\not{l}\gamma_\nu\right]\right)$$

First two terms also cancel after shifts.

We get:

$$(k_1+k_2)^\rho\,T_{\mu\nu\rho} = -e^2\int\frac{d^4l}{(2\pi)^4}\,m^2\left\{\frac{1}{\left[l^2-m^2\right]\left[(l+k_1)^2-m^2\right]\left[(l-k_2)^2-m^2\right]}\right.$$

$$\cdot\left\{\mathrm{Tr}\left[\gamma_5\gamma_\mu\not{l}\gamma_\nu(\not{l}-\not{k}_2)\right]+\mathrm{Tr}\left[\gamma_5(\not{l}+\not{k}_1)\gamma_\mu\not{l}\gamma_\nu\right]-\right.$$

$$-\mathrm{Tr}\left[(\not{k}_1+\not{k}_2)\gamma_5\gamma_\mu\gamma_\nu(\not{l}-\not{k}_2)\right]-\mathrm{Tr}\left[(\not{k}_1+\not{k}_2)\gamma_5\cdot(\not{l}+\not{k}_1)\gamma_\mu\gamma_\nu\right]-\mathrm{Tr}\left[(\not{k}_1+\not{k}_2)\gamma_5\gamma_\mu\not{l}\gamma_\nu\right]\left.\vphantom{\Big[}\right\}+\begin{pmatrix}k_1\leftrightarrow k_2\\ \mu\leftrightarrow\nu\end{pmatrix}$$

m^2 terms in Tr.

$$=\left(\text{as } \mathrm{Tr}\left[\gamma_5\gamma^\mu\gamma^\nu\gamma^\alpha\gamma^\beta\right]=-4i\,\varepsilon^{\mu\nu\alpha\beta}\right) =$$

$$= 4ie^2\,\varepsilon^{\mu\nu\alpha\beta}\int\frac{d^4l}{(2\pi)^4}\;\frac{m^2}{(l^2-m^2)\left[(l+k_1)^2-m^2\right]\left[(l-k_2)^2-m^2\right]}$$

$$\Big\{-l_\alpha(\cancel{l}-k_2)_\beta + l_\alpha(\cancel{l}+k_1)_\beta-(l-k_2)_\alpha(k_1+k_2)_\beta+(k_1+k_2)_\alpha(l+k_1)_\beta$$

$$-(k_1+k_2)_\alpha\, l_\beta\Big\}_{+\left(\substack{k_1\leftrightarrow k_2\\ \mu\leftrightarrow\nu}\right)} = 4i\,e^2\,\varepsilon^{\mu\nu\alpha\beta}\int\frac{d^4l}{(2\pi)^4}\cdot$$

$$\frac{m^2}{\left[l^2-m^2\right]\left[(l+k_1)^2-m^2\right]\left[(l-k_2)^2-m^2\right]}\Big\{\cancel{l_\alpha k_{2\beta}+l_\alpha k_{1\beta}}\;-$$

$$-\cancel{l_\alpha(k_1+k_2)_\beta}+\cancel{(k_1+k_2)_\alpha l_\beta}-\cancel{(k_1+k_2)_\alpha l_\beta}+k_{2\alpha}(k_1+\underbrace{\cancel{k_2}}_{0})_\beta$$

$$+k_{1\beta}(\underbrace{\cancel{k_1}}_{0}+k_2)_\alpha\Big\}_{+\left(\substack{\mu\leftrightarrow\nu\\ k_1\leftrightarrow k_2}\right)} = 8\,i\,e^2\,\varepsilon^{\mu\nu\alpha\beta}\,k_{1\beta}\,k_{2\alpha}\cdot\underbrace{\int\frac{d^4l}{(2\pi)^4}\cdot}$$

$$\cdot\underbrace{\frac{m^2}{(l^2-\underset{+i\varepsilon}{m^2})\left[(l+k_1)^2-\underset{+i\varepsilon}{m^2}\right]\left[(l-k_2)^2-\underset{+i\varepsilon}{m^2}\right]}}_{I_m} + \left(\substack{k_1\leftrightarrow k_2\\ \mu\leftrightarrow\nu}\right)$$

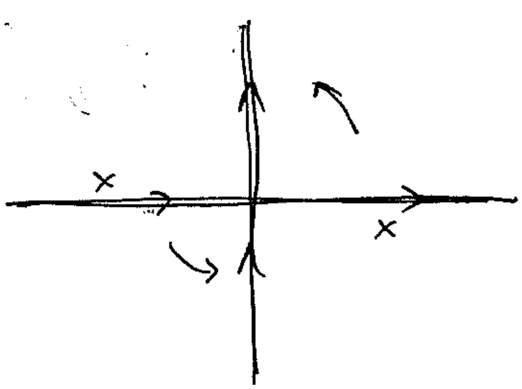
$k_1, k_2 \ll m$ always

if $l \ll m \Rightarrow I_m \sim \frac{1}{m^4}\to 0$

if $l \gtrsim m \Rightarrow l, m \gg k_1, k_2$

Approximate the integral by: ($l, m \sim$ large)

$$I_m \approx \int\frac{d^4l}{(2\pi)^4}\;\frac{m^2}{\left[l^2-m^2+i\varepsilon\right]^3} = \Bigg|\;\begin{array}{l}\text{Wick rotation}\\ l_0 = +i\,l_0^E\end{array}$$



$$l^2-m^2+i\varepsilon = \left(l_0-\sqrt{\vec{l}^{\,2}+m^2}+i\varepsilon\right)\cdot\left(l_0+\sqrt{\vec{l}^{\,2}+m^2}-i\varepsilon\right)$$