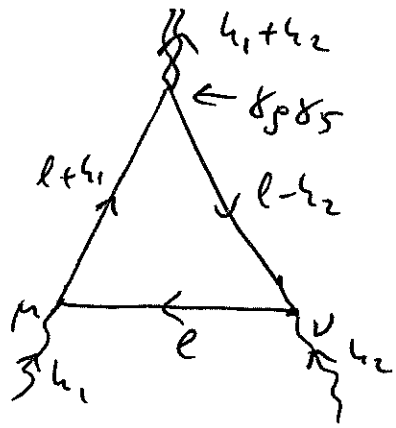
<u>Last time</u> Calculated triangle diagrams:

$T_{\mu\nu\rho}(k_1,k_2) =$  $+ \begin{pmatrix} \mu \leftrightarrow \nu \\ k_1 \leftrightarrow k_2 \end{pmatrix}$

After some algebra we've arrived at

$$(k_1+k_2)^\rho\, T_{\mu\nu\rho}(k_1,k_2) = 8\, i\, e^2\, \varepsilon^{\mu\nu\alpha\beta}\, k_{1\beta}\, k_{2\alpha} \cdot \int \frac{d^4 l}{(2\pi)^4} \frac{m^2}{[l^2-m^2+i\varepsilon][(l+k_1)^2-m^2+i\varepsilon][(l-k_2)^2-m^2+i\varepsilon]} + \begin{pmatrix} k_1 \leftrightarrow k_2 \\ \mu \leftrightarrow \nu \end{pmatrix}$$

$\Rightarrow$ the $m=0$ (first) term in [...] vanishes like before.

for the term in [...] containing m^2 write:

$$\mathrm{Tr}\left[\left(\not{k}_1+\not{l}-(\not{l}-\not{k}_2)\right)\gamma_5(\not{l}+\not{k}_1)\gamma_\mu\not{l}\gamma_\nu(\not{l}-\not{k}_2)\right] =$$

$$= -(l+k_1)^2\,\mathrm{Tr}\left[\gamma_5\gamma_\mu\not{l}\gamma_\nu(\not{l}-\not{k}_2)\right]-(l-k_2)^2\,\mathrm{Tr}\left[\gamma_5(\not{l}+\not{k}_1)\cdot\gamma_\mu\not{l}\gamma_\nu\right] = -\left[(l+k_1)^2-m^2\right]\mathrm{Tr}\left[\gamma_5\gamma_\mu\not{l}\gamma_\nu(\not{l}-\not{k}_2)\right]$$

$$-\left[(l-k_2)^2-m^2\right]\mathrm{Tr}\left[\gamma_5(\not{l}+\not{k}_1)\gamma_\mu\not{l}\gamma_\nu\right]-m^2\left(\mathrm{Tr}\left[\gamma_5\gamma_\mu\not{l}\gamma_\nu\cdot(\not{l}-\not{k}_2)\right]+\mathrm{Tr}\left[\gamma_5(\not{l}+\not{k}_1)\gamma_\mu\not{l}\gamma_\nu\right]\right)$$

First two terms also cancel after shifts.

We get:

$$(k_1+k_2)^\rho T_{\mu\nu\rho} = -e^2\int\frac{d^4l}{(2\pi)^4}\, m^2 \left\{ \frac{1}{\left[l^2-m^2\right]\left[(l+k_1)^2-m^2\right]\left[(l-k_2)^2-m^2\right]}\right.$$

$$\cdot\left\{\mathrm{Tr}\left[\gamma_5\gamma_\mu\not{l}\gamma_\nu(\not{l}-\not{k}_2)\right]+\mathrm{Tr}\left[\gamma_5(\not{l}+\not{k}_1)\gamma_\mu\not{l}\gamma_\nu\right]-\right.$$

$$-\mathrm{Tr}\left[(\not{k}_1+\not{k}_2)\gamma_5\gamma_\mu\gamma_\nu(\not{l}-\not{k}_2)\right]-\mathrm{Tr}\left[(\not{k}_1+\not{k}_2)\gamma_5\cdot(\not{l}+\not{k}_1)\gamma_\mu\gamma_\nu\right]-\mathrm{Tr}\left[(\not{k}_1+\not{k}_2)\gamma_5\gamma_\mu\not{l}\gamma_\nu\right]\Big\} + \begin{pmatrix}k_1\leftrightarrow k_2\\ \mu\leftrightarrow\nu\end{pmatrix}\Bigg\}_{m^2 \text{ terms in Tr.}}$$

$$= \left(\text{as } \mathrm{Tr}\left[\gamma^5\gamma^\mu\gamma^\nu\gamma^\alpha\gamma^\beta\right] = -4i\,\varepsilon^{\mu\nu\alpha\beta}\right) =$$

$$= 4 i e^2 \varepsilon^{\mu\nu\alpha\beta} \int \frac{d^4 l}{(2\pi)^4} \frac{m^2}{(l^2-m^2)\left[(l+k_1)^2-m^2\right]\left[(l-k_2)^2-m^2\right]}$$

$$\cdot \Big\{ -l_\alpha (\cancel{l}-k_2)_\beta + l_\alpha(\cancel{l}+k_1)_\beta - (l-k_2)_\alpha (k_1+k_2)_\beta + (k_1+k_2)_\alpha (l+k_1)_\beta$$

$$- (k_1+k_2)_\alpha l_\beta \Big\}_{+\binom{k_1\leftrightarrow k_2}{\mu\leftrightarrow\nu}} = 4ie^2 \varepsilon^{\mu\nu\alpha\beta}\int\frac{d^4l}{(2\pi)^4}\cdot$$

$$\cdot \frac{m^2}{[l^2-m^2]\left[(l+k_1)^2-m^2\right]\left[(l-k_2)^2-m^2\right]} \Big\{ \cancel{l_\alpha k_{2\beta}} + \cancel{l_\alpha k_{1\beta}} -$$

$$- \cancel{l_\alpha(k_1+k_2)_\beta} + \cancel{(k_1+k_2)_\alpha l_\beta} - \cancel{(k_1+k_2)_\alpha l_\beta} + k_{2\alpha}(k_1+\underbrace{k_2}_{0})_\beta$$

$$+ k_{1\beta}(\underbrace{k_1}_{0}+k_2)_\alpha \Big\}_{+\binom{\mu\leftrightarrow\nu}{k_1\leftrightarrow k_2}} = 8 i e^2 \varepsilon^{\mu\nu\alpha\beta} k_{1\beta} k_{2\alpha} \cdot \int\frac{d^4l}{(2\pi)^4}\cdot$$

$$\cdot \underbrace{\frac{m^2}{(l^2-m^2+i\varepsilon)\left[(l+k_1)^2-m^2+i\varepsilon\right]\left[(l-k_2)^2-m^2+i\varepsilon\right]}}_{I_m} + \binom{k_1\leftrightarrow k_2}{\mu\leftrightarrow\nu}$$

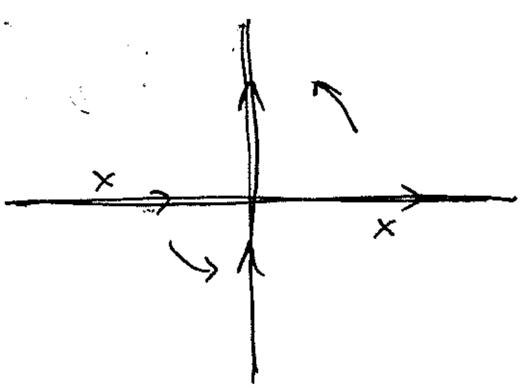
$k_1, k_2 \ll m$ always

if $l \ll m \Rightarrow I_m \sim \frac{1}{m^4} \to 0$

if $l \gtrsim m \Rightarrow l, m \gg k_1, k_2$

Approximate the integral by: ($l, m \sim$ large)

$$I_m \simeq \int \frac{d^4 l}{(2\pi)^4} \frac{m^2}{\left[l^2-m^2+i\varepsilon\right]^3} = \Big| \text{ Wick rotation } l_0 = +i\, l_0^E$$



$$l^2 - m^2 + i\varepsilon = \left(l_0 - \sqrt{\vec{l}^{\,2}+m^2} + i\varepsilon\right) \cdot \left(l_0 + \sqrt{\vec{l}^{\,2}+m^2} - i\varepsilon\right)$$

$\Rightarrow I_m = -i\int \frac{d^4 l_E}{(2\pi)^4}\, \frac{m^2}{\left[l_E^2+m^2\right]^3} = -i\int_0^\infty \frac{l_E^3\, dl_E}{(2\pi)^4}\cdot \underbrace{\int d\Omega_4}_{2\pi^2}\cdot$

$\cdot\frac{m^2}{\left[l_E^2+m^2\right]^3} = -i\,\frac{1}{8\pi^2}\, m^2 \int_0^\infty \frac{dl\cdot l^3}{\left[l^2+m^2\right]^3} = -i\,\frac{1}{16\pi^2}\, m^2\cdot$

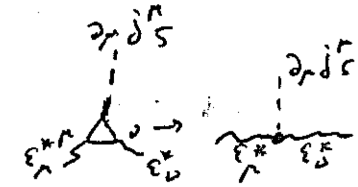
$\cdot\int_0^\infty \frac{dl^2\cdot\left[l^2+m^2-m^2\right]}{\left[l^2+m^2\right]^3} = -i\,\frac{1}{(4\pi)^2}\, m^2\cdot\left[\frac{1}{m^2} - m^2\,\frac{1}{2m^4}\right] =$

$= -i\,\frac{1}{2}\,\frac{1}{(4\pi)^2}.$ We get

$\begin{pmatrix}\mu\leftrightarrow\nu\\ k_1\leftrightarrow k_2\end{pmatrix}$ $\downarrow$

$(k_1+k_2)^\rho\, T_{\mu\nu\rho} = 8\cancel{i}e^2\, \varepsilon^{\mu\nu\alpha\beta}\, k_{1\beta}\, k_{2\alpha}\, \cancel{(-i)}\,\frac{1}{2}\,\frac{1}{(4\pi)^2}\, 2 \Rightarrow$

$(k_1+k_2)^\rho\, T_{\mu\nu\rho} = -2\frac{\alpha_{EM}}{\pi}\, \varepsilon^{\mu\nu\alpha\beta}\, k_{1\alpha}\, k_{2\beta}$



$\Rightarrow$ in operator language this means ($\frac{1}{2}$ symm. factor)

$$\boxed{\partial_\mu j_5^\mu = \frac{-\alpha}{4\pi}\, \varepsilon^{\mu\nu\alpha\beta}\, F_{\mu\nu} F_{\alpha\beta}}$$

Adler-Bell-Jackiw anomaly '69

$\Rightarrow$ classically conserved current is <u>not</u> conserved quantum mechanically!

$\Rightarrow$ in QED this ABJ anomaly relation is exact ~ no higher-order corrections.

In QCD have $j_{5\mu} = \sum_f \bar{q}_f \gamma_\mu \gamma_5 q_f$

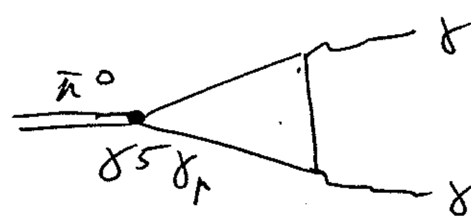
and $$\boxed{\partial_\mu j_5^\mu = -\frac{\alpha_s N_f}{8\pi} \varepsilon^{\mu\nu\alpha\beta} F^a_{\mu\nu} F^a_{\rho\sigma}}.$$

$\Rightarrow$ $U(1)_A$ in QCD is broken, but has no Goldstone boson associated with this breaking $\Rightarrow$ symmetry was never there in the full quantum theory

(Otherwise, if treating $U(1)_A$ as a symmetry, would expect parity-doubling of baryon states. If $U(1)_A$ is broken $\sim$ expect Goldstone modes. This way we see that the symmetry is never a good symmetry.)

$\Rightarrow$ to get $\neq 0$ $\varepsilon^{\mu\nu\alpha\beta} F^a_{\mu\nu} F^a_{\rho\sigma}$ need instantons ...

$\Rightarrow$ axial anomaly is responsible for pion decay: $\pi^0 \to \gamma\gamma$



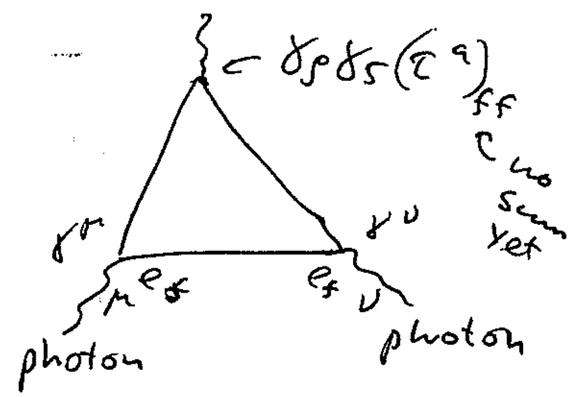
$\pi^0 \to \gamma\gamma$

in QCD with $N_f = 2$

Consider axial isospin current $j^a_{5\mu} = \bar{q}\gamma_\mu\gamma_5\tau^a q$

where τ^a = Pauli matrices, $a = 1, 2, 3$ (flavor index for $SU(2)$ flavor). Here $q = \begin{pmatrix} u \\ d \end{pmatrix}$.

It has an anomaly due to quarks coupling to photons:



$$\partial_\mu j_5^{a\mu} = -\frac{\alpha_{EM}}{4\pi}\varepsilon^{\alpha\beta\mu\nu}F_{\alpha\beta}F_{\mu\nu} \cdot \sum_f (\tau^a)_{ff}\cdot e_f^2$$

as $\tau^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $\tau^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$, $\tau^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

$\Rightarrow$ only τ^3 gives $\neq 0$ anomaly

$$\sum_f (\tau^3)_{ff}\, e_f^2 = \left(\frac{2}{3}\right)^2 - \left(\frac{1}{3}\right)^2 = \frac{1}{3}$$

$$\Rightarrow \partial_\mu j_5^{3\mu} = -\frac{\alpha_{EM}}{12\pi}\varepsilon^{\mu\nu\alpha\beta}F_{\mu\nu}F_{\alpha\beta}$$

$\pi^0 = \frac{\bar{u}u - \bar{d}d}{\sqrt{2}}$.

$j_5^{3\mu} = \bar{u}\gamma_\mu\gamma_5 u - \bar{d}\gamma_\mu\gamma_5 d$ annihilates (or creates) π^0:

$$\langle 0 | j_5^{3\mu}(0) | \pi^0(p)\rangle = i f_\pi p^\mu$$

(due to spont. chiral symm. breaking)

axial charge does not annihil vac

with $f_\pi \simeq 93$ MeV (pion decay constant)

$\Rightarrow$ in general $\langle 0 | j_5^{3\mu}(x) | \pi^0(p) \rangle = i p^\mu f_\pi e^{-ip\cdot x}$

$$\Rightarrow \langle 0 | \partial_\mu j_5^{3\mu}(x) | \pi^0(p) \rangle = \underbrace{p_\mu p^\mu}_{m_\pi^2} f_\pi e^{-ip\cdot x}$$

$$\Rightarrow \langle 0 | \partial_\mu j_5^{3\mu}(0) | \pi^0(p) \rangle = m_\pi^2 f_\pi$$

$\Rightarrow$ pion couples to $\partial_\mu j_5^{3\mu} \sim \varepsilon^{\mu\nu\alpha\beta} F_{\mu\nu} F^{\alpha\beta}$

$\Rightarrow \sim A_\rho A_\sigma \Rightarrow$ pion couples to two photons

$\Rightarrow$ can have $\pi^0 \to \gamma\gamma$ decay due to the axial anomaly.

Axial anomaly in the Standard Model.

$\Rightarrow$ a theory with axial anomaly would violate Ward identities $\left((k_1+k_2)^\rho T_{\mu\nu\rho}=0\right)$, an is therefore not gauge invariant!

$\Rightarrow$ this would be a problem for theories with axial current coupling to gauge bosons (e.g. SM)

$\Rightarrow$ in particular an anomaly would spoil renormalizability of the theory

$\Rightarrow$ Standard Model has vector bosons coupling with γ_5 to leptons and quarks. For SM to be consistent need those 3-boson couplings with γ_5 to cancel!

Let's go back to SM Lagrangian:

$$\boxed{\begin{aligned}\mathcal{L} &= \bar{R}_e\, i\gamma^\mu\left(\partial_\mu + \frac{ig'}{2} Y B_\mu\right) R_e + \bar{L}_e\, i\gamma^\mu\left(\partial_\mu + i\frac{g'}{2} Y B_\mu - ig\frac{\vec{\tau}}{2}\cdot\vec{W}_\mu\right) L_e + (\mu,\tau) + \bar{L}_u\, i\gamma^\mu\left(\partial_\mu + i\frac{g'}{2} Y B_\mu - ig\frac{\vec{\tau}}{2}\cdot\vec{W}_\mu\right) \\ &\cdot L_u + \bar{R}_u\, i\gamma^\mu\left(\partial_\mu + i\frac{g'}{2} Y B_\mu\right) R_u + \bar{R}_d\, i\gamma^\mu\left(\partial_\mu + i\frac{g'}{2} Y B_\mu\right) R_d \\ &+ (2\text{ more generations}) + \ldots\end{aligned}}$$

(we keep quark/lepton-vector boson terms only)

Y is the weak hypercharge

$$\boxed{Q = I_3 + \frac{Y}{2}}$$ Gell-Mann – Nishijima relation always holds.

⇒ for L_e : $I_3 = \pm 1/2$; $Q = 0$ for neutrino

$$\Rightarrow \quad 0 = \frac{1}{2} + \frac{Y}{2} \Rightarrow Y_{L_e} = -1$$

for R_e have $I_3 = 0$, $Q = -1 \Rightarrow -1 = \frac{Y}{2} \Rightarrow Y_{R_e} = -2$

for L_u: u-quark has $Q = +2/3 \Rightarrow \frac{2}{3} = \frac{1}{2} + \frac{Y}{2}$

$$\Rightarrow Y_{L_u} = 1/3$$

for R_u: $I_3 = 0 \Rightarrow \frac{2}{3} = \frac{Y}{2} \Rightarrow Y_{R_u} = 4/3$

for R_d: $Q = -1/3 \Rightarrow -1/3 = \frac{Y}{2} \Rightarrow Y_{R_d} = -2/3$

other generations ~ same ⇒ forget about them.

as $L_e = \begin{pmatrix} \nu_e \\ e \end{pmatrix}_L = \frac{1-\gamma_5}{2}\begin{pmatrix} \nu_e \\ e \end{pmatrix}$, $R_e = \frac{1+\gamma_5}{2} e = e_R$

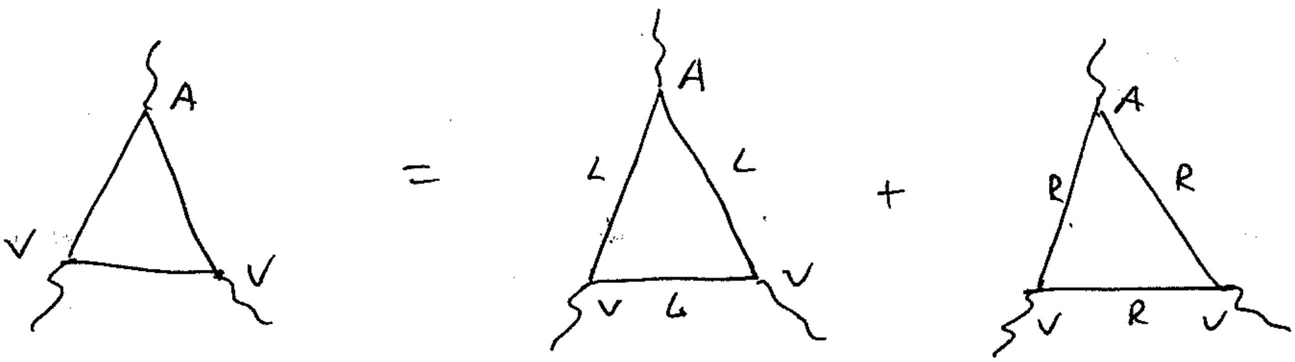
⇒ all W_μ, B_μ couplings involve γ_5 ⇒ need divergences to cancel.

massless QED

$$\mathcal{L}_{QED} = \bar\psi\, i\gamma^\mu D_\mu \psi - \frac{1}{4} F_{\mu\nu}F^{\mu\nu} = \bar\psi_L\, i\gamma^\mu D_\mu \psi_L$$

$$+ \bar\psi_R\, i\gamma^\mu D_\mu \psi_R - \frac{1}{4} F_{\mu\nu}F^{\mu\nu}$$

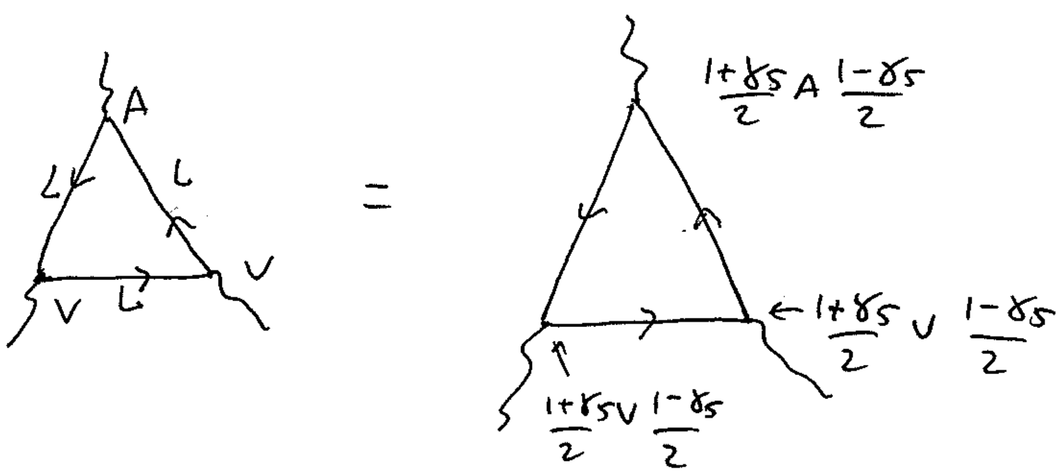
⇒ the anomaly consists of left-handed and right-handed (massless) electrons' contributions



$A = \gamma_\rho \gamma_5$

$V = \gamma_\mu$, or γ_ν

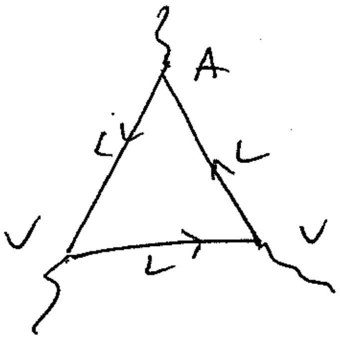
Propagator $\langle \psi_L \bar\psi_L \rangle = \langle \frac{1-\gamma_5}{2} \psi \bar\psi \frac{1+\gamma_5}{2} \rangle$ ⇒



$$\frac{1+\gamma_5}{2}\gamma_\mu \frac{1-\gamma_5}{2} = \gamma_\mu \frac{1-\gamma_5}{2} = \frac{V-A}{2}$$

$$\frac{1+\gamma_5}{2}\gamma_\rho\gamma_5 \frac{1-\gamma_5}{2} = \gamma_\rho\gamma_5 \frac{1-\gamma_5}{2} = \gamma_\rho \frac{\gamma_5 - 1}{2} = \frac{A-V}{2}$$

Hence



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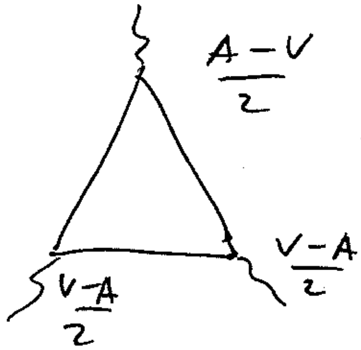


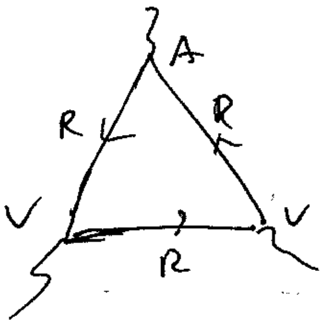
$= \frac{1}{2}$ Anomaly

Subtract, get

$= 0$

$\Rightarrow$ anomalies cancel!

Similarly



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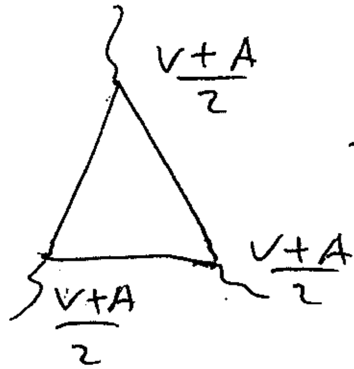
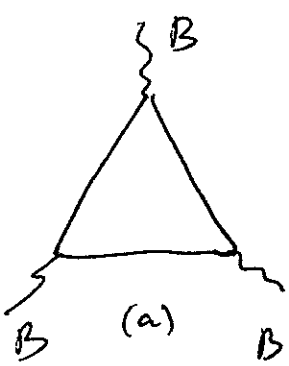


$= \frac{1}{2}$ Anomaly

No anomaly in 3-boson coupling! ($\approx$ QED)

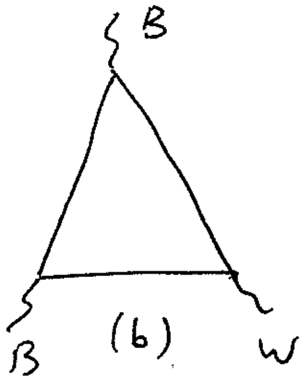
$\Rightarrow$ in SM need to sum all graphs with left- and right-handed particles in the loop.

The diagrams are:



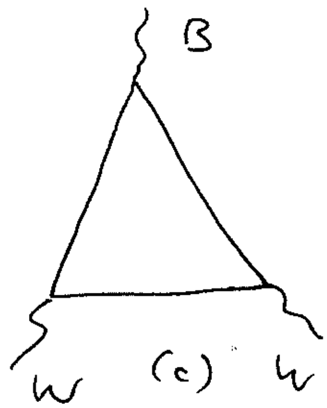
(a)

,

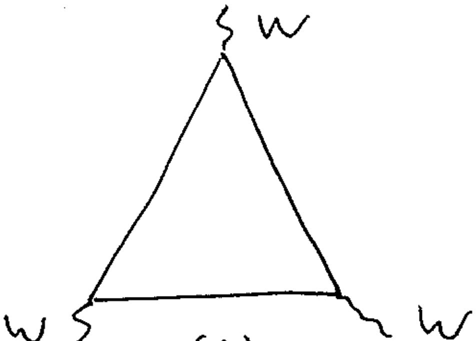


(b)

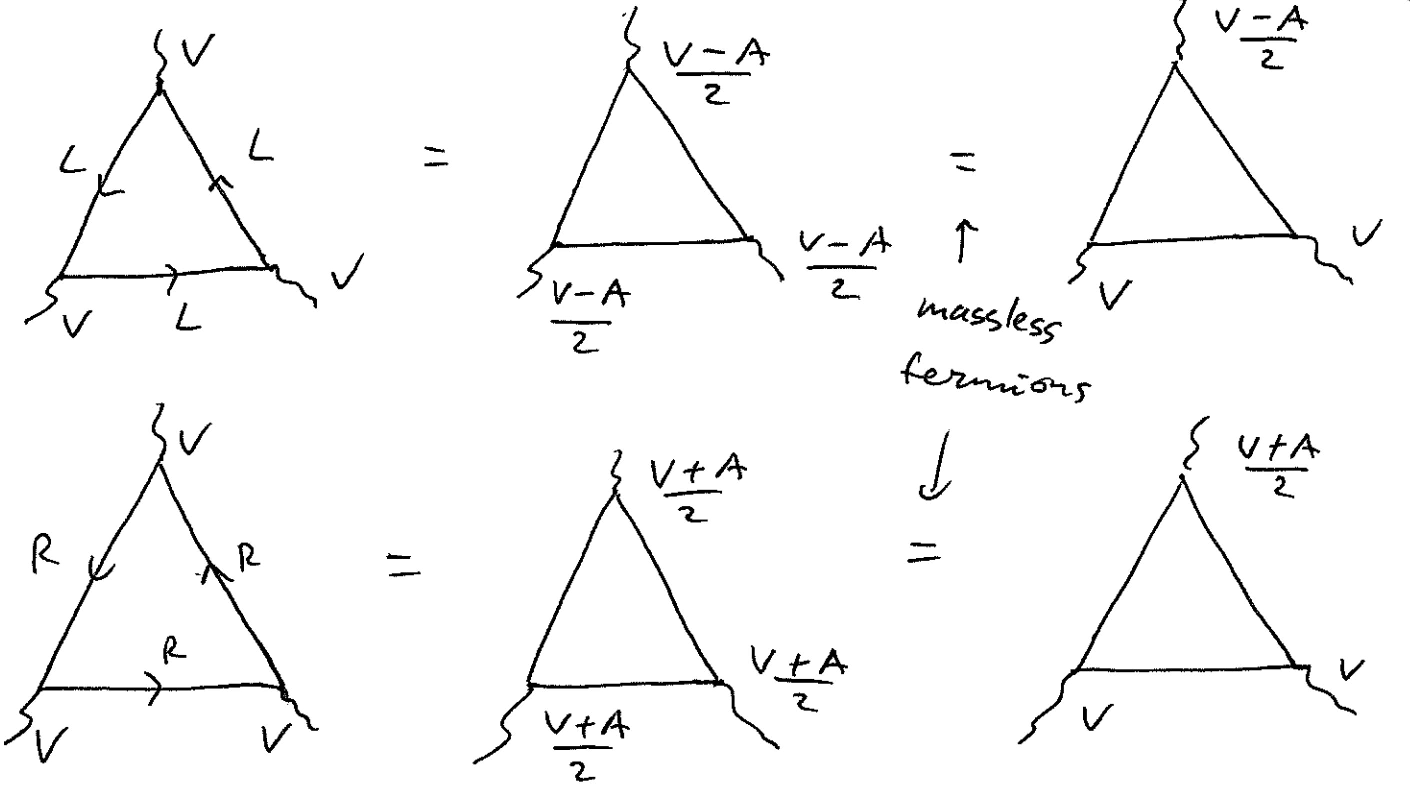
,



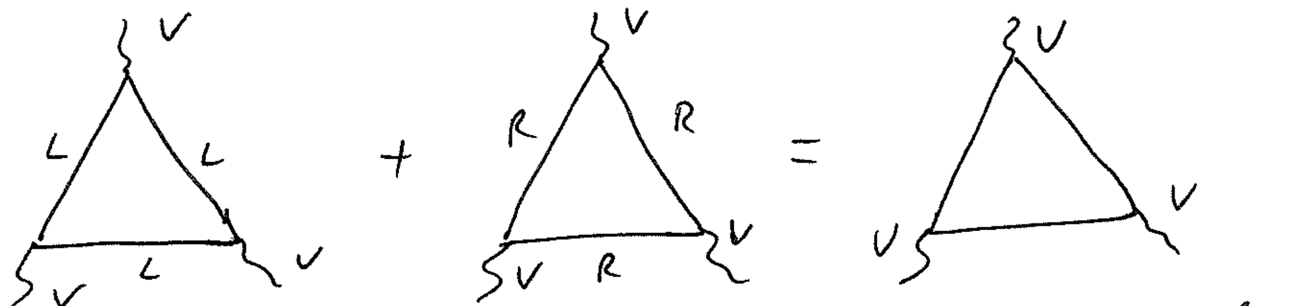
(c)



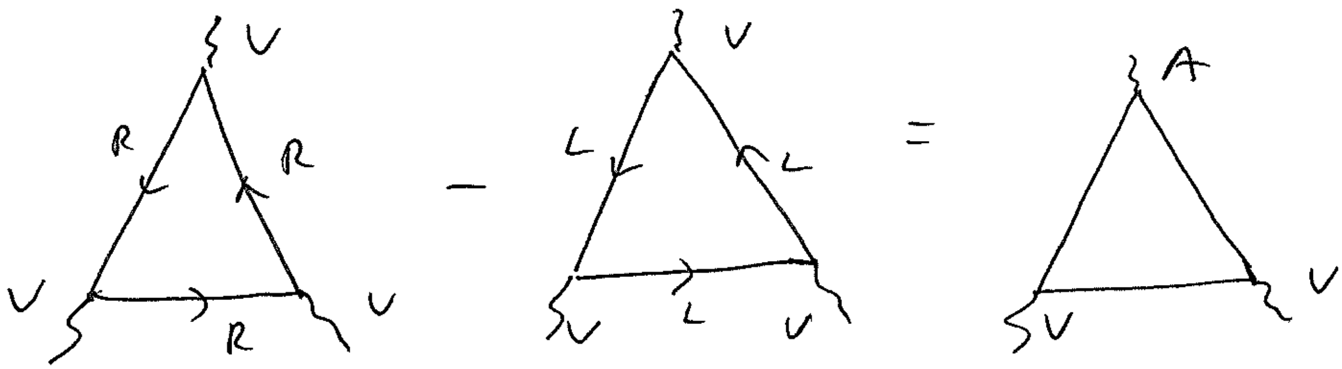
(d)



=>



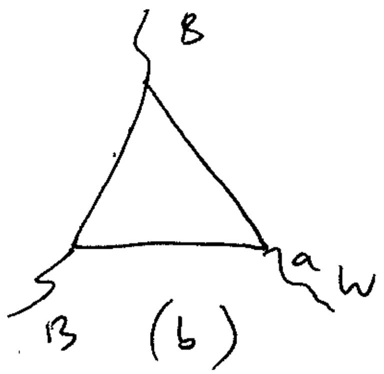
as expected



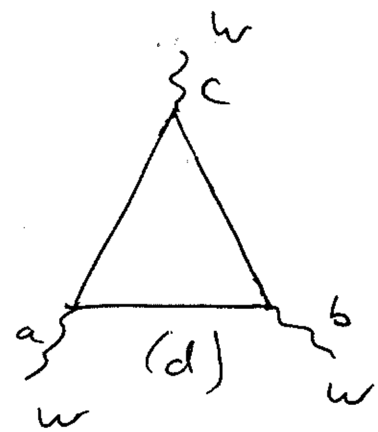
the anomaly.

Anomaly is the difference between the right-handed and left-handed loops.

First let's do (b): $\mathrm{tr}\,\tau^a = 0 \Rightarrow \boxed{(b) = 0.}$



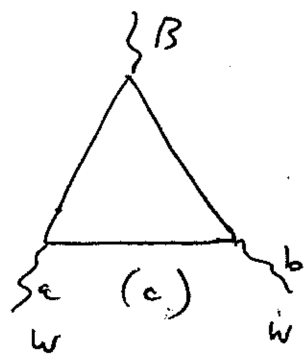
Now, let's look at (d):



$$\mathrm{tr}\left(\tau^c \tau^a \tau^b\right) + \mathrm{tr}\left(\tau^c \tau^b \tau^a\right)$$

$$= \mathrm{tr}\left[\tau^c \underbrace{\{\tau^a, \tau^b\}}_{2\delta^{ab}}\right] \sim \mathrm{tr}\,\tau^c = 0 \Rightarrow \boxed{(d) = 0}$$

Next let's look at (c):



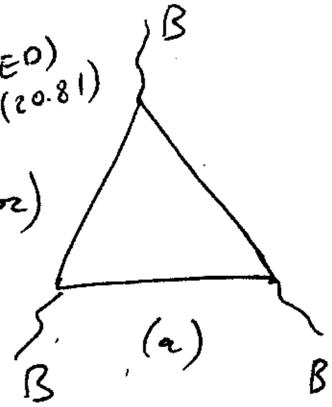
$\mathrm{tr}\,\frac{\tau^a}{2}\frac{\tau^b}{2} = \frac{1}{2}\delta^{ab}$ is not zero

$$(c) \propto \sum_{\substack{i = \text{left-handed} \\ \text{doublets}}} Y_i$$ (as W couples to left-handed quarks & leptons only)

$$\Rightarrow (c) \propto Y_{Le} + Y_{Lu} \cdot \underset{\uparrow \text{ No. of colors}}{3} = -1 + \frac{1}{3}\cdot 3 = 0$$

$$\Rightarrow \boxed{(c) = 0}$$

Finally, let's look at (a):



contribute "–" to anomaly (see QED) Peskin (20.81)

$$(a) \propto 2 \sum_{i = \text{left-handed doublets}} Y_i^3 \times (\text{color}) - \sum_{i = \text{right-handed}} Y_i^3 \otimes (\text{color})$$

doublets

$$= 2\,(-1)^3 + 2 \cdot \left(\tfrac{1}{3}\right)^3 \cdot 3 - (-2)^3 - \left(\tfrac{4}{3}\right)^3 \cdot 3 - \left(-\tfrac{2}{3}\right)^3 \cdot 3$$

L_e $\quad L_u$ color $\quad R_e$ $\quad R_u$ color $\quad R_d$ color

$$= -2 + \frac{2}{9} + 8 - \frac{64}{9} + \frac{8}{9} = 6 - \frac{54}{9} = 0$$

$$\Rightarrow \boxed{(a) = 0}.$$

$\Rightarrow$ the same applies to the other two generations

$\Rightarrow$ anomalies cancel in 3-vector boson couplings in the SM! Thus Standard Model is a consistent (gauge-invariant) and renormalizable theory... as expected.