Non-Abelian Gauge Theories

~ we will consider theories with $SU(N)$ local gauge symmetry.

~ to construct Lagrangian for such theories start with $U(1)$ symmetry

Abelian Gauge Theories (brief review)

take Dirac field: $\mathcal{L} = \bar{\psi}\left[i\not{\partial} - m\right]\psi$

$\Rightarrow$ $\mathcal{L}$ is invariant under a global $U(1)$ symmetry: $\psi \to e^{i\alpha}\psi$, $\bar{\psi} \to \bar{\psi}e^{-i\alpha}$

α ~ real number

$\Rightarrow$ make it local: require that the Lagrangian has local $U(1)$ symmetry:

$$\psi(x) \to e^{i\alpha(x)}\psi(x), \quad \bar{\psi}(x) \to \bar{\psi}(x)e^{-i\alpha(x)}$$

$$\bar{\psi}\,[i\gamma\cdot\partial - m]\,\psi \to \bar{\psi}\, e^{-i\alpha(x)}\,[i\gamma\cdot\partial - m]\, e^{i\alpha(x)}\psi$$

$$= \bar{\psi}\,[\,i\gamma\cdot\partial + i\gamma\cdot\partial(i\alpha) - m\,]\psi = \bar{\psi}\,[i\gamma\cdot\partial - m]\psi$$

$$- \bar{\psi}\,\gamma^\mu(\partial_\mu\alpha)\,\psi$$

$\Rightarrow$ $\mathcal{L}$ is not invariant under local $U(1)$ symmetry!

$\Rightarrow$ Fix it by introducing local gauge field $A_\mu(x)$ (gauge the Lagrangian)

$$\mathcal{L} = \bar{\psi}\,[\,i\gamma\cdot\partial - m\,]\,\psi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} - e\bar{\psi}\gamma^\mu A_\mu\psi$$

$\Rightarrow$ require that: $\boxed{\begin{array}{l}\psi \to e^{i\alpha(x)}\psi \\ A_\mu \to A_\mu - \frac{1}{e}\partial_\mu\alpha(x)\end{array}}$

$$\Rightarrow \mathcal{L} \to \mathcal{L}' = \bar{\psi}\,[\,i\gamma\cdot\partial - m\,]\psi - \cancel{\bar{\psi}\gamma^\mu(\partial_\mu\alpha)\psi} - \frac{1}{4}F_{\mu\nu}F^{\mu\nu}$$

$$+ e\bar{\psi}\gamma^\mu A_\mu\psi + \cancel{\bar{\psi}\gamma^\mu(\partial_\mu\alpha)\psi} = \mathcal{L}$$

$\Rightarrow$ now it is invariant!

$\Rightarrow$ Def. Covariant derivative $D_\mu \equiv \partial_\mu + ieA_\mu$

$$\Rightarrow \mathcal{L}_{QED} = \bar{\psi}\,[\,i\gamma^\mu D_\mu - m\,]\,\psi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu}$$

as usual $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$, $F_{\mu\nu} = [D_\mu D_\nu - D_\nu D_\mu]\frac{-i}{e} = -\frac{i}{e}[D_\mu, D_\nu]$

Now imagine a $\underline{\text{SU(2) Gauge Theory.}}$ theory with a non-abelian symmetry, like $SU(2)$: $\psi = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}$, $\psi_1, \psi_2 \sim$ spinors

ψ_1 & ψ_2 are different by some quantum #
(e.g. color, weak isospin)

$\mathcal{L} = \bar{\psi}\left[i\gamma^\mu \partial_\mu - m\right]\psi$ with $m = \begin{pmatrix} m & 0 \\ 0 & m \end{pmatrix}$ is

invariant under $\psi \to \psi' = e^{i\vec{\alpha}\cdot\frac{\vec{\sigma}}{2}}\psi$

$\vec{\sigma}$ are Pauli matrices in $\begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}$ space.

$\Rightarrow$ <u>global</u> $SU(2)$ symmetry.

$\Rightarrow$ let's make it <u>local</u> (gauge it): $\vec{\alpha} = \vec{\alpha}(x)$

$$\Rightarrow \psi \to \psi' = e^{i\vec{\alpha}(x)\cdot\frac{\vec{\sigma}}{2}}\psi(x) \equiv S(x)\,\psi(x)$$

with $S^+ S = S S^+ = \mathbb{1}$.

$$\Rightarrow \mathcal{L} \to \mathcal{L}' = \bar{\psi}\, S^+\left[i\gamma^\mu \partial_\mu - m\right] S\,\psi = \bar{\psi}\left[i\gamma\cdot\partial - m\right]\psi$$

$$+ \bar{\psi}\, i\gamma^\mu \left(S^+ \partial_\mu S\right)\psi \quad \Rightarrow \text{not invariant}$$

$\Rightarrow$ add a gauge field A_μ^a, $a = 1, 2, 3$:

$$\boxed{\mathcal{L} = \bar{\psi}\left[i\gamma\cdot\partial - m\right]\psi - \frac{1}{4}F_{\mu\nu}^a F^{a\mu\nu} + g\,\bar{\psi}\gamma^\mu A_\mu^a \frac{\sigma^a}{2}\psi}$$

$$\mathcal{L} \to \mathcal{L}' = \bar{\Psi}\left[i\gamma^\mu \partial_\mu - m\right]\Psi + \bar{\Psi}\, i\gamma^\mu \left(S^+ \partial_\mu S\right)\Psi +$$

$$+ g\bar{\Psi}\gamma^\mu S^+ A'_\mu S \Psi - \frac{1}{4} F'^{a}_{\mu\nu} F'^{a\mu\nu}$$

where $A_\mu = A^a_\mu \dfrac{\sigma^a}{2}$ is a matrix.

Collect Ψ-terms: $g\bar{\Psi}\gamma^\mu \underbrace{\left[S^+ A'_\mu S + \frac{i}{g} S^+ \partial_\mu S\right]}_{\text{require } = A_\mu}\Psi$

$$\Rightarrow A_\mu = S^+ A'_\mu S + \frac{i}{g} S^+ \partial_\mu S \quad \Rightarrow S A_\mu S^+ = A'_\mu +$$

$$+ \frac{i}{g}(\partial_\mu S) S^+ \Rightarrow \begin{aligned} A'_\mu &= S A_\mu S^+ - \frac{i}{g}(\partial_\mu S) S^+ \\ \Psi' &= S\Psi \end{aligned}$$

non-Abelian gauge transformation!

Def. Covariant derivative $D_\mu = \partial_\mu - igA_\mu$

(note: now it's a matrix!)

$$\Rightarrow \boxed{\mathcal{L} = \bar{\Psi}\left[i\gamma^\mu D_\mu - m\right]\Psi - \frac{1}{4} F^a_{\mu\nu} F^{a\mu\nu}}$$

<u>But</u>: we never checked the invariance of $F^a_{\mu\nu} F^{a\mu\nu}$ term. What is $F^a_{\mu\nu}$ anyway? Using Abelian analogy write $F_{\mu\nu} = \frac{i}{g}\left[D_\mu, D_\nu\right]$

where $F_{\mu\nu} = F_{\mu\nu}^a \frac{\sigma^a}{2}$.

$$F_{\mu\nu} = \frac{i}{g}\left[D_\mu, D_\nu\right] = \frac{i}{g}\left[\partial_\mu - igA_\mu, \partial_\nu - igA_\nu\right] =$$

$$= \frac{i}{g}\left\{ -ig\left[\partial_\mu, A_\nu\right] - ig\left[A_\mu, \partial_\nu\right] - g^2\left[A_\mu, A_\nu\right]\right\}$$

$$= \frac{i}{g}\left\{ -ig\left(\partial_\mu A_\nu - \partial_\nu A_\mu\right) - g^2\left[A_\mu, A_\nu\right]\right\} =$$

$$= \partial_\mu A_\nu - \partial_\nu A_\mu - ig\left[A_\mu, A_\nu\right]$$

$$\Rightarrow \quad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - ig\left[A_\mu, A_\nu\right]$$

$$F_{\mu\nu}^a \frac{\sigma^a}{2} = \left(\partial_\mu A_\nu^a - \partial_\nu A_\mu^a\right)\frac{\sigma^a}{2} - ig A_\mu^b A_\nu^c \underbrace{\left[\frac{\sigma^b}{2}, \frac{\sigma^c}{2}\right]}_{i\varepsilon^{bca}\frac{\sigma^a}{2}}$$

← SU(2)

$$= \frac{\sigma^a}{2}\left[\partial_\mu A_\nu^a - \partial_\nu A_\mu^a + g\,\varepsilon^{abc} A_\mu^b A_\nu^c\right]$$

$$\Rightarrow \quad F_{\mu\nu}^a = \partial_\mu A_\nu^a - \partial_\nu A_\mu^a + g\,\varepsilon^{abc} A_\mu^b A_\nu^c$$

~ true for SU(2)

~ other groups have different group structure constants:

$$F_{\mu\nu}^a = \partial_\mu A_\nu^a - \partial_\nu A_\mu^a + g f^{abc} A_\mu^b A_\nu^c.$$

What happens to $F_{\mu\nu}$ under non-Abelian gauge transform?

Start with D_μ: $D_\mu = \partial_\mu - igA_\mu \rightarrow$

$$\rightarrow \partial_\mu - ig\left[S A_\mu S^{-1} - \frac{i}{g}(\partial_\mu S) S^{-1} \right] =$$

$$= S\left[\partial_\mu - igA_\mu \right] S^{-1} = S D_\mu S^{-1}$$

as $S \partial_\mu S^{-1} = \partial_\mu + S(\partial_\mu S^{-1})$

now: $\mathbb{1} = S S^{-1} \Rightarrow 0 = \partial_\mu (S S^{-1}) = (\partial_\mu S) S^{-1} + S(\partial_\mu S^{-1})$

$\Rightarrow S(\partial_\mu S^{-1}) = -(\partial_\mu S) S^{-1} \Rightarrow S \partial_\mu S^{-1} = \partial_\mu - (\partial_\mu S) S^{-1}$

$$\Rightarrow \quad D_\mu \rightarrow S D_\mu S^{-1}$$

$$\Rightarrow \quad F_{\mu\nu} = \frac{i}{g}[D_\mu, D_\nu] \rightarrow \frac{i}{g}[S D_\mu S^{-1}, S D_\nu S^{-1}]$$

$$= \frac{i}{g} S [D_\mu, D_\nu] S^{-1} = S F_{\mu\nu} S^{-1}$$

$$\Rightarrow \quad F_{\mu\nu} \rightarrow F'_{\mu\nu} = S F_{\mu\nu} S^{-1}$$

$\Rightarrow$ Note that $F_{\mu\nu}$ is <u>not</u> invariant under gauge transformation if it is non-Abelian!

$$-\frac{1}{4} F^a_{\mu\nu} F^{a\mu\nu} = -\frac{1}{2} \operatorname{tr}\left(F_{\mu\nu} F^{\mu\nu}\right)$$

as $\operatorname{tr}\left(\frac{\sigma^a}{2}\frac{\sigma^b}{2}\right) = \frac{1}{2}\delta^{ab}$ $\Rightarrow$ under non-Abelian gauge transformation have

$$-\frac{1}{2}\operatorname{tr}\left(F_{\mu\nu}F^{\mu\nu}\right) \to -\frac{1}{2}\operatorname{tr}\left(F'_{\mu\nu}F'^{\mu\nu}\right) = -\frac{1}{2}\operatorname{tr}\left[S' F_{\mu\nu} \cancel{S'^{-1}} \cancel{S'} F^{\mu\nu} \cancel{S'^{-1}}\right] = -\frac{1}{2}\operatorname{tr}\left[F_{\mu\nu}F^{\mu\nu}\right]$$

$\Rightarrow$ the Lagrangian is invariant under non-Abelian gauge transformation:

$$\boxed{\mathcal{L} = \bar{\psi}\left[i\gamma^\mu D_\mu - m\right]\psi - \frac{1}{2}\operatorname{tr}\left(F_{\mu\nu}F^{\mu\nu}\right)}$$

true for any gauge group $SU(N)$

$$D_\mu = \partial_\mu - igA_\mu, \quad F_{\mu\nu} = \frac{i}{g}\left[D_\mu, D_\nu\right].$$

The Higgs Mechanism (U(1) model)

~ Imagine a case when gauge symmetry is spontaneously broken

~ Goldstone Th'm does not apply: needs manifest Lorentz invariance (VEV is $\mathcal{L}$ inv) & positivity of the norm. (to have G. boson state w/ $\langle u|u\rangle > 0$)

In gauge theories $\mathcal{L}$. inv. gauges $\partial_\mu A^\mu = 0$ don't have >0 of the norm, other gauges $A^0 = 0$, $\vec{\nabla}\cdot\vec{A} = 0$ are not

Generalization: SU(N) Gauge Theory

For a SU(N) gauge theory use $N \times N$ generators of su(N) in the fundamental representation T^a:

$$[T^a, T^b] = i f^{abc} T^c$$

$\uparrow$ structure constants

$$\Rightarrow A_\mu = A_\mu^a T^a \quad , \quad a = 1, \dots, N^2 - 1$$

$F_{\mu\nu} = F_{\mu\nu}^a T^a \Rightarrow$ again $F_{\mu\nu} = \frac{i}{g}[D_\mu, D_\nu]$ with the covariant derivative $D_\mu = \partial_\mu - ig A_\mu$. One can show that

$$\boxed{F_{\mu\nu}^a = \partial_\mu A_\nu^a - \partial_\nu A_\mu^a + g f^{abc} A_\mu^b A_\nu^c}.$$

The gauge-invariant Lagragian is then:

$$\boxed{\mathcal{L} = \bar{\psi}[i\gamma^\mu D_\mu - m]\psi - \frac{1}{4} F_{\mu\nu}^a F^{a\mu\nu}}.$$

ψ^i, $i = 1, \dots, N$ ~ N different spinors

$A_\mu' = S A_\mu S^{-1} - \frac{i}{g}(\partial_\mu S) S^{-1}$, $\psi' = S\psi$ ~ gauge transform

$$D_\mu \to S D_\mu S^{-1}$$

$\Rightarrow \mathcal{L}$ is invariant under SU(N) local gauge symmetry!

Quantum Chromodynamics (QCD): theory of quarks and gluons. SU(3) gauge group

Quark fields: q_α^{if} ← color, $i=1,2,3$; ← flavor index, $f=u,d,s,c,b,t$

↑ spinor index $\alpha = 1,2,3,4$

A_μ^a ~ gluon fields

color, $a=1,\dots,8$

Lorentz index $\mu = 0,1,2,3$

The Lagrangian is

$$\boxed{\mathcal{L}_{QCD} = \bar{q}^{if}\left[i\gamma\cdot D_{ij} - m_f\right] q^{jf} - \frac{1}{4} F_{\mu\nu}^a F^{a\mu\nu}}$$

$D_\mu = \partial_\mu - i g A_\mu^a T^a$, $\quad T^a = \frac{\lambda^a}{2}$ ~ Gell-Mann matrices

⇒ Sum over flavors and colors assumed.

⇒ Other local non-Abelian theories in nature: electroweak interactions (SU(2) group).