Functional Quantization of the Scalar Field Theory. (put $\hbar=1$ again)

Again use the analogy between $q \leftrightarrow \varphi(x)$, $p \leftrightarrow \pi(x)$, $L(q,\dot{q}) \leftrightarrow \mathcal{L}(\varphi, \partial_\mu \varphi)$ => replace

$$\int [\mathcal{D}q] \to \int [\mathcal{D}\varphi]$$

$$S = \int dt\, L(q,\dot{q}) \to S = \int d^4x\, \mathcal{L}(\varphi, \partial_\mu \varphi)$$

=> introduce generating functional by

$$\boxed{Z[j(x)] = \int \mathcal{D}\varphi\, e^{i\int d^4x\,[\mathcal{L} + j(x)\varphi(x)]}}$$

$$\mathcal{H} = \frac{1}{2}\left(\pi^2 + (\vec{\nabla}\varphi)^2\right) + \frac{m^2}{2}\varphi^2 + \frac{\lambda}{4!}\varphi^4 \quad \text{for } \varphi^4 \text{ theory}$$

$$\mathcal{L} = \frac{1}{2}\partial_\mu\varphi\,\partial^\mu\varphi - \frac{m^2}{2}\varphi^2 - \frac{\lambda}{4!}\varphi^4.$$

$${}_H\langle \varphi_f, t_f | \varphi_i, t_i \rangle_H = \int \mathcal{D}\varphi\, \mathcal{D}\pi\, e^{i\int d^4x\,[\pi\dot{\varphi} - \mathcal{H}]} =$$

$$= \int \mathcal{D}\varphi\, e^{i\int d^4x\, \mathcal{L}}$$

as $$\boxed{\langle \psi_0 | T\{\varphi_H(x_1) \ldots \varphi_H(x_n)\} | \psi_0 \rangle = \frac{\int \mathcal{D}\varphi\, \varphi(x_1)\ldots\varphi(x_n)\, e^{i\int d^4x\,\mathcal{L}}}{\int \mathcal{D}\varphi\, e^{i\int d^4x\,\mathcal{L}}}}$$

=> $$\boxed{\langle \psi_0 | T\, \varphi_H(x_1)\ldots\varphi_H(x_n) | \psi_0 \rangle = (-i)^n \frac{1}{Z[0]} \frac{\delta^n Z[j]}{\delta j(x_1)\ldots\delta j(x_n)}\bigg|_{j=0}}$$

$\Rightarrow$ generating functional generates all possible n-point functions.

Free scalar theory

$$\mathcal{L}_0 = \frac{1}{2}\partial_\mu\varphi\,\partial^\mu\varphi - \frac{m^2}{2}\varphi^2 \quad \Rightarrow$$

picks out vacua at $t = \pm\infty$

$$Z_0[j] = \int \mathcal{D}\varphi\; e^{i\int d^4x\,[\frac{1}{2}\partial_\mu\varphi\,\partial^\mu\varphi - \underbrace{\frac{m^2 - i\varepsilon}{2}}\varphi^2 + j\varphi]}$$

$$= (\text{parts}) = \int \mathcal{D}\varphi\; e^{-i\int d^4x\,[\frac{1}{2}\varphi(\partial_\mu\partial^\mu + \underset{-i\varepsilon}{m^2})\varphi - j\varphi]}$$

Gaussian integrals: $\displaystyle\int_{-\infty}^{\infty} dx\, e^{-\frac{a}{2}x^2} = \sqrt{\frac{2\pi}{a}}$.

$$\int_{-\infty}^{\infty} dx_1\ldots dx_n\; e^{-\frac{a_1}{2}x_1^2 - \ldots - \frac{a_n}{2}x_n^2} = \frac{(2\pi)^{n/2}}{\sqrt{a_1 a_2 \ldots a_n}}.$$

Define an $n\times n$ matrix $A = \begin{pmatrix} a_1 & & 0 \\ & a_2 & \\ 0 & & \ddots\; a_n \end{pmatrix}$

$$x = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \Rightarrow \boxed{\int d^n x\; e^{-\frac{1}{2}x^T A x} = \frac{(2\pi)^{n/2}}{\sqrt{\det A}}}$$

True for any real symmetric matrix A, since can always diagonalize: $A' = S A S^{-1}$, such

orthogonal, $S^{-1} = S^T$.

diagonal

that $\det A' = \cancel{\det S} \cdot \det A \, \cancel{\det S^{-1}} = \det A$.

$y = S' x \Rightarrow d^4y = \underbrace{|\det S'|}_{=1} d^4x$

Defining $(dx) = d^4x \, (2\pi)^{-4/2}$ get | as S' is orthogonal ~~(or unitary)~~.

$$\boxed{\int (dx) \, e^{-\frac{1}{2} x^T A x} = \frac{1}{\sqrt{\det A}}}$$

Similarly, for functional integrals

$$\int \mathcal{D}\varphi \, e^{-\frac{1}{2}\int d^4x \, \varphi(x) \overset{\text{some operator}}{\hat{D}} \varphi(x)} = \frac{1}{\sqrt{\det \hat{D}}}.$$

We see that

$$Z_0[j] = \int \mathcal{D}\varphi \, e^{-i\int d^4x \left[\frac{1}{2}\varphi(\Box + m^2_{i\varepsilon})\varphi - j\varphi\right]}$$

$\Rightarrow$ write $\frac{1}{2}\varphi\underbrace{(\Box + m^2_{i\varepsilon})}_{\hat{D}}\varphi - j\varphi = \frac{1}{2}\underbrace{(\varphi - j\hat{D}^{-1})}_{\text{"}\tilde{\varphi}^T\text{"}}\hat{D}\underbrace{(\varphi - \hat{D}^{-1}j)}_{\tilde{\varphi}}$

$- \frac{1}{2} j \hat{D}^{-1} j \quad \Rightarrow$

$$Z_0[j] = \int \mathcal{D}\tilde{\varphi} \, e^{-i\int d^4x \left[\frac{1}{2}\tilde{\varphi}\hat{D}\tilde{\varphi} - \frac{1}{2} j \hat{D}^{-1} j\right]}$$

$$\Rightarrow \boxed{Z_0[j] = \frac{1}{\sqrt{\det(i\hat{D})}} \, e^{i\int d^4x \frac{1}{2} j \hat{D}^{-1} j}}$$

$\hat{D} = \Box + m^2 \underbrace{- i\varepsilon}_{\text{picks out the right vacuum.}}$

Explanation: more Gaussian integrals:

$$I \equiv \int (dx)\, e^{-\frac{1}{2} x^T A x + J^T \cdot x}$$

$$J = \begin{pmatrix} J^1 \\ \vdots \\ J^n \end{pmatrix} \sim \text{a "vector"}, \qquad J^T \cdot x = x^T J$$

$$\Rightarrow I = \int (dx)\, e^{-\frac{1}{2} \underbrace{(x^T - J^T A^{-1})}_{\tilde{x}^T} A \underbrace{(x - A^{-1} J)}_{\tilde{x}} + \frac{1}{2} J^T A^{-1} J}$$

$$= \frac{1}{\sqrt{\det A}} \cdot e^{\frac{1}{2} J^T A^{-1} J}.$$

To find $\hat{D}^{-1}$ we write the integral differently:

$$Z_0[j] = \int \mathcal{D}\varphi\, e^{-i\int d^4x \left[\frac{1}{2}\varphi(\Box + m^2 - i\varepsilon)\varphi - j\varphi\right]} = \Big/ \varphi \to \varphi + \varphi_0$$

such that $(\Box + m^2 - i\varepsilon)\phi_0 = j \quad \Rightarrow$

$$Z_0[j] = \int \mathcal{D}\varphi\, e^{-i\int d^4x \left[\frac{1}{2}\varphi(\Box + m^2 - i\varepsilon)\varphi + \frac{1}{2}\varphi_0(\Box + m^2 - i\varepsilon)\varphi\right.}$$

$$\left. + \frac{1}{2}\varphi(\Box + m^2 - i\varepsilon)\varphi_0 + \frac{1}{2}\varphi_0(\Box + m^2 - i\varepsilon)\varphi_0 - j\varphi - j\varphi_0\right] = (\text{parts, etc})$$

$$= \int \mathcal{D}\varphi\, e^{-i\int d^4x \left[\frac{1}{2}\varphi\underbrace{(\Box + m^2 - i\varepsilon)}_{\hat{D}}\varphi - \frac{1}{2}\varphi_0 \cdot j\right]} =$$

$$= \frac{1}{\sqrt{\det(i\hat{D})}} \cdot e^{\frac{i}{2}\int d^4x\, j \cdot \varphi_0}.$$

$\Rightarrow (\Box + m^2 - i\varepsilon)\,\varphi_0 = j$ $\Rightarrow$ start by noting that $(\Box_x + m^2 - i\varepsilon)\, D_F(x-y) = -i\,\delta^{(4)}(x-y)$

with $$\boxed{D_F(x-y) = \int \frac{d^4p}{(2\pi)^4}\, e^{-ip\cdot(x-y)}\, \frac{i}{p^2 - m^2 + i\varepsilon}}$$

$$\Rightarrow \boxed{\varphi_0(x) = i\int d^4y\; D_F(x-y)\, j(y)} = \hat{D}^{-1} j$$

$$\Rightarrow \boxed{Z_0[j] = \frac{1}{\sqrt{\det(\hat{D})}} \cdot e^{-\frac{1}{2}\int d^4x\, d^4y\; j(x)\, D_F(x-y)\, j(y)}}$$

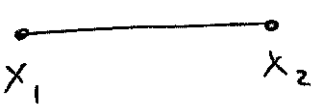
(btw $\hat{D}^{-1} = i\int d^4y\; D_F(x-y)$).

$\nwarrow$ "inverse propagator"

$$\Rightarrow \underset{\langle 0|T\varphi(x_1)\varphi(x_2)|0\rangle}{\underset{\parallel}{\langle \psi_0 | T\varphi_H(x_1)\,\varphi_H(x_2)|\psi_0\rangle_{\text{free}}}} = (-i)^2 \frac{1}{Z_0(0)}\, \frac{\delta^2 Z_0[j]}{\delta j(x_1)\,\delta j(x_2)}\Bigg|_{j=0}$$

$$= (-)\,\frac{\delta^2}{\delta j(x_1)\,\delta j(x_2)} \left[e^{-\frac{1}{2}\int d^4x\, d^4y\; j(x)\, D_F(x-y)\, j(y)} \right]\Bigg|_{j=0}$$

$= D_F(x_1 - x_2)$ $\Rightarrow$ get correct propagator!



(used the fact that $D_F(x-y) = D_F(y-x)$)

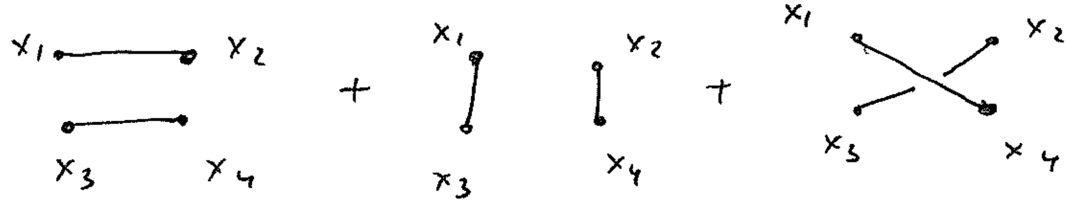
$\Rightarrow$ One may also calculate higher order Green functions: $\underset{\langle 0|T\varphi(x_1)\varphi(x_2)\varphi(x_3)\varphi(x_4)|0\rangle}{\underset{\parallel}{\langle\psi_0|T\varphi(x_1)\varphi(x_2)\varphi(x_3)\varphi(x_4)|\psi_0\rangle_{\text{free}}}} =$

$$= (-i)^4 \frac{1}{Z_0[0]} \frac{\delta^4 Z_0[j]}{\delta j(x_1)\,\delta j(x_2)\,\delta j(x_3)\,\delta j(x_4)}\Bigg|_{j=0} =$$

$$= \frac{\delta^4}{\delta j(x_1)\dots\delta j(x_4)}\left\{ e^{-\frac{1}{2}\int d^4x\, d^4y\; j(x)\, D_F(x-y)\, j(y)} \right\}\Bigg|_{j=0}$$

$$= D_F(x_1-x_2)\,D_F(x_3-x_4) + D_F(x_1-x_3)\,D_F(x_2-x_4) +$$

$$+ D_F(x_1-x_4)\,D_F(x_2-x_3).$$



just like before!

φ^4 theory

For the interacting scalar theory with

$$\mathcal{L} = \frac{1}{2}\partial_\mu\varphi\,\partial^\mu\varphi - \frac{m^2}{2}\varphi^2 - \frac{\lambda}{4!}\varphi^4$$

one has

$$\boxed{Z[j] = \int \mathcal{D}\varphi \cdot e^{i\int d^4x\left[\frac{1}{2}\partial_\mu\varphi\,\partial^\mu\varphi - \frac{m^2-i\varepsilon}{2}\varphi^2 - \frac{\lambda}{4!}\varphi^4 + j\cdot\varphi\right]}}$$

$\Rightarrow$ this is not a Gaussian integral so it is <u>hard</u> to integrate over φ analytically (try $\int_{-\infty}^{\infty} dx\, e^{-ax^4-bx^2}$).

Instead we write

$$Z[j] = e^{i\int d^4x\left(\frac{-\lambda}{4!}\right)\cdot\left(-i\frac{\delta}{\delta j}\right)^4}\cdot\int\mathcal{D}\varphi\, e^{i\int d^4x\left[\frac{1}{2}\partial_\mu\varphi\partial^\mu\varphi - \frac{m^2-i\varepsilon}{2}\varphi^2 + j\varphi\right]} \Rightarrow \boxed{Z[j] = e^{-i\frac{\lambda}{4!}\int d^4x\frac{\delta^4}{\delta j^4}}Z_0[j]}$$

$\Rightarrow$ Can expand perturbatively in λ $\Rightarrow$ obtain Feynman diagrams and perturbation theory.

Consider 2-point function $\langle\Psi_0|T\,\varphi(x_1)\varphi(x_2)|\Psi_0\rangle$:

In general

$$\langle\Psi_0|T\,\varphi(x_1)\varphi(x_2)|\Psi_0\rangle = \frac{1}{Z[0]}(-i)^2\cdot\frac{\delta^2 Z[j]}{\delta j(x_1)\delta j(x_2)}\Bigg|_{j=0}$$

$$= -\frac{\left\{\frac{\delta^2}{\delta j(x_1)\delta j(x_2)}e^{-i\frac{\lambda}{4!}\int d^4x\frac{\delta^4}{\delta j^4}}Z_0[j]\right\}\Big|_{j=0}}{\left\{e^{-i\frac{\lambda}{4!}\int d^4x\frac{\delta^4}{\delta j^4}}Z_0[j]\right\}\Big|_{j=0}} =$$

$$= -\frac{\left\{\frac{\delta^2}{\delta j(x_1)\delta j(x_2)}e^{-i\frac{\lambda}{4!}\int d^4x\frac{\delta^4}{\delta j(x)^4}}\cdot e^{-\frac{1}{2}\int d^4y\,d^4z\, j(y)D_F(y-z)j(z)}\right\}\Big|_{j=0}}{\left\{e^{-i\frac{\lambda}{4!}\int d^4x\frac{\delta^4}{\delta j(x)^4}}\cdot e^{-\frac{1}{2}\int d^4y\,d^4z\, j(y)D_F(y-z)j(z)}\right\}\Big|_{j=0}}$$

at order-λ^0 just get free theory

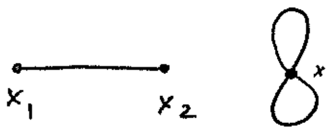
result: $\langle\psi_0|T\varphi(x_1)\varphi(x_2)|\psi_0\rangle\Big|_{O(\lambda^0)} = D_F(x_1-x_2).$

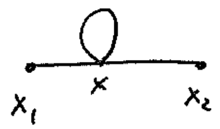
at order-λ get:

$$\text{Numerator} = i\frac{\lambda}{4!}\left\{\frac{\delta^2}{\delta j(x_1)\delta j(x_2)}\int d^4x\,\frac{\delta^4}{\delta j(x)^4}\,e^{-\frac{1}{2}\int d^4y\,d^4z\,j(y)D_F(y-z)j(z)}\right\}\Bigg|_{j=0}$$

$\int d^4x$

expand the exponent

$= i\frac{\lambda}{4!}\left\{\frac{1}{3!}\frac{-1}{2^3}\left[3\cdot2\cdot4!\right.\right.$ 

$+\ 3\cdot2\cdot2^2\cdot4!$ 

$\left.\left.\right]\right\}$

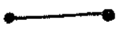
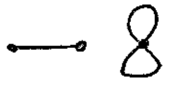
$= -i\lambda\int d^4x\left[\frac{1}{8}\right.$  $+\ \frac{1}{2}$  $\left.\right]$

$$\text{DENOMINATOR} = \left[1 - i\frac{\lambda}{4!}\int d^4x\,\frac{\delta^4}{\delta j(x)^4}\right]e^{-\frac{1}{2}\int d^4y\,d^4z\,j(y)D_F(y-z)j(z)}\Bigg|_{j=0}$$

$= 1 - i\frac{\lambda}{\cancel{4!}}\int d^4x\cdot\frac{1}{2!}\left(-\frac{1}{2}\right)^2\cancel{4!}$ 

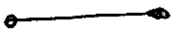
$= 1 - i\lambda\int d^4x\ \frac{1}{8}$ 

$\Rightarrow$ get $\langle\psi_0|T\varphi(x_1)\varphi(x_2)|\psi_0\rangle =$

$=$  $- i\lambda\int d^4x\left[\frac{1}{8}\right.$  $+\ \frac{1}{2}$  $\left.\right] + \cdots$

over

$1 - i\lambda\int d^4x\ \frac{1}{8}$  $+ \cdots$

$=$  $- i \lambda \int d^4x \ \frac{1}{2}$  $+ \dots$

$\Rightarrow$ again the denominator cancels all the disconnected graphs!

(Can prove this to all orders similar to the canonical quantization case.)

$\Rightarrow$ We see that we can build the Feynman rules and perturbation theory: they are identical to what we had before.

$\Rightarrow$ For general interaction scalar theory with

$$\mathcal{L} = \frac{1}{2} \partial_\mu \varphi \partial^\mu \varphi - \frac{m^2}{2} \varphi^2 + \mathcal{L}_{int}(\varphi)$$

write

$$\boxed{Z[j] = e^{i\int d^4x \, \mathcal{L}_{int}\left(-i\frac{\delta}{\delta j}\right)} Z_0[j]}$$

and expand in $\mathcal{L}_{int}$.

n-point functions are given by

$$\boxed{\langle \Psi_0 | T \, \varphi(x_1) \dots \varphi(x_n) | \Psi_0 \rangle = \frac{1}{Z[0]} (-i)^n \frac{\delta^n Z[j]}{\delta j(x_1) \dots \delta j(x_n)} \Big|_{j=0}}$$

Finally, let's normalize $Z[j]$ to be 1 at $j=0$, i.e., take $\frac{Z[j]}{Z[0]}$ and write

Def. $$\frac{Z[j]}{Z[0]} = e^{iW[j]}$$

$W[j]$ is the generating functional of connected Green functions.

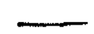
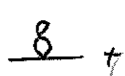
$$W[j] = -i \ln\{Z[j]/Z[0]\}$$

$$\Rightarrow \quad \frac{\delta W[j]}{\delta j(x_1)\,\delta j(x_2)} = -i \frac{\delta}{\delta j(x_1)}\left[\frac{1}{Z[j]}\cdot\frac{\delta Z[j]}{\delta j(x_2)}\right] =$$

$$= -i\frac{1}{Z[j]}\frac{\delta^2 Z[j]}{\delta j(x_1)\delta j(x_2)} + i\frac{1}{Z^2[j]}\frac{\delta Z[j]}{\delta j(x_1)}\frac{\delta Z[j]}{\delta j(x_2)}$$

In φ^4 theory (or ∀ scalar theory) have $\left.\frac{\delta Z}{\delta j}\right|_{j=0} = 0 \Rightarrow$

$$\left.\frac{\delta W[j]}{\delta j(x_1)\delta j(x_2)}\right|_{j=0} = -i\frac{1}{Z[0]}\left.\frac{\delta^2 Z[j]}{\delta j(x_1)\,\delta j(x_2)}\right|_{j=0} = i\,D_F(x_1-x_2)+\ldots$$

$= i\langle \Psi_0 | T\,\varphi(x_1)\,\varphi(x_2)|\Psi_0\rangle = i[$  $+$  $+$  $+ \;]$

"connected" really connected. means no vacuum bubbles here & no graphs like , etc.

$\Rightarrow$ also works for higher order Green functions.