Feynman Rules and Cross Sections

⇒ to perform calculations in the interacting field theory need to construct perturbation theory using Feynman rules.

⇒ non-perturbative methods exist (e.g. lattice, effective theories), but often difficult and sometimes impossible

⇒ as $\alpha_{EM} = \frac{e^2}{4\pi} \simeq \frac{1}{137} \ll 1$, $\frac{g^2}{4\pi} \simeq \frac{1}{30} \ll 1$ ⇒

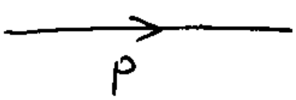
⇒ can construct pert. theory in them!

Consider QED: $\mathcal{L}_{QED} = \bar{\psi}\, i\gamma\cdot\partial\psi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} - e\bar{\psi}\gamma\cdot A\psi$

a theory of electrons ψ & photons A_μ.

Feynman Rules for QED

electron propagator $\qquad \dfrac{i}{\not{p} - m + i\varepsilon}$ (assigned to each internal electron line)



$\not{p} = \gamma^\mu p_\mu$.

photon propagator

$$\frac{-i}{k^2 + i\varepsilon}\left[g_{\mu\nu} + (\xi - 1)\frac{k_\mu k_\nu}{k^2}\right],$$

$\xi = 1$ Feynman gauge, $\xi = 0$ Landau gauge

(assigned to each internal photon line)

electron-photon vertex:

$-ie\gamma^\mu$

(in more complicated theory get other factors on top of γ^μ)

External photon lines:

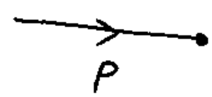
incoming: 

$\Rightarrow \varepsilon_\mu^\lambda(k)$, λ→polarization

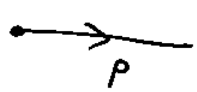
out going: 

$\Rightarrow \varepsilon_\mu^{\lambda *}(k)$

External fermions:

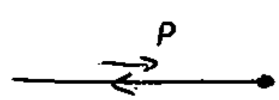
incoming: 

$\Rightarrow u_s(p)$

outgoing: 

$\Rightarrow \bar{u}_s(p)$

External anti-fermions:

incoming: 

$\Rightarrow \bar{v}_s(p)$

(note that momentum flows in, fermion # flows out)

outgoing: 

$\Rightarrow v_s(p)$

Cross section = $\dfrac{\text{event prod. per unit volume \& time}}{(\text{incident flux}) \times (\text{target density})}$

$$|\text{final state}\rangle = \mathcal{S}\, |\text{initial state}\rangle$$

$= |\psi_f\rangle$, $= |\psi_i\rangle$

Def. $\longrightarrow$ $\mathcal{S}$ - matrix (time-evolution operator)

$$|\psi_f\rangle = \mathcal{S}\,|\psi_i\rangle = \left[\mathbb{1} + (\mathcal{S}-1)\right]|\psi_i\rangle = |\psi_i\rangle + $$
$$+ (\mathcal{S}-1)\,|\psi_i\rangle$$

$\Rightarrow$ the interaction is in $S-1$ $\Rightarrow$

Def. T-matrix is defined by $\boxed{\mathcal{S} = \mathbb{1} + i\,\mathbb{T}}$

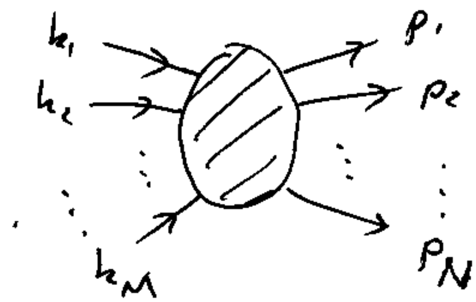
Unitarity: $\mathcal{S}^+ \mathcal{S} = \mathbb{1} \Rightarrow \mathbb{1} - iT^+ + iT + T^+T = \mathbb{1}$

$\Rightarrow\; i\,(T - T^+) = -T^+T \Rightarrow 2\,\mathrm{Im}\,T = T^+T$ $\Rightarrow$ optical theorem

Def. <u>Scattering amplitude:</u>

$\Rightarrow$ initial state is $|k_1, k_2, .., k_M\rangle$

final (outgoing) free-particle state $|p_1, \dots, p_N\rangle$

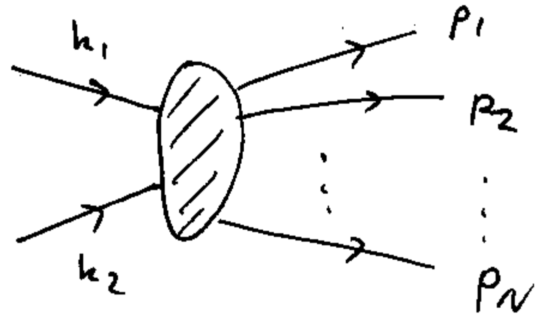


$\Rightarrow$ transition amplitude M is defined by

$$\langle p_1, \dots, p_N|\, \mathcal{S} - 1\, |k_1, \dots, k_M\rangle = i \langle p_1, \dots, p_N|\, T\, |k_1, \dots, k_M\rangle \equiv$$

$$\equiv (2\pi)^4\, \delta^{(4)}(p_1 + \dots + p_N - k_1 - \dots - k_M)\; i\, M(k_1, \dots, k_M; p_1, \dots, p_N)$$

For simplicity consider $2 \to N$ process:



$\Rightarrow$ one can show that the <u>cross section</u> for the process is:

$$\boxed{d\sigma = \frac{1}{2\mathcal{E}_{k_1}\, 2\mathcal{E}_{k_2}\, |\vec{v}_1 - \vec{v}_2|} \prod_{i=1}^{N} \frac{d^3 p_i}{(2\pi)^3\, 2\mathcal{E}_i} \left| M(k_1, k_2; p_1, \ldots, p_N) \right|^2 \cdot (2\pi)^4\, \delta^{(4)}\Big(k_1 + k_2 - \sum_{j=1}^{N} p_j\Big)}$$

(see Peskin & Schroeder, Ryder, ...)

$\vec{v}_1, \vec{v}_2$ ~ 3-velocities of the incoming particles.

Is this object Lorentz-invariant?

$$\frac{d^3 p_i}{(2\pi)^3\, 2\mathcal{E}_i} = \int \frac{d^4 p_i}{(2\pi)^4}\, 2\pi\, \delta(p_i^2 - m_i^2)\, \theta(p_i^0) \sim \text{Lorentz-inv.}$$

$\delta^{(4)}$ is $\mathcal{L}$-inv. (why?)

$|M|^2$ is $\mathcal{L}$ inv. (see definition)

What about $\mathcal{E}_{k_1} \mathcal{E}_{k_2} |\vec{v}_1 - \vec{v}_2|$? Note that $v_i = \frac{\vec{k}_i}{\mathcal{E}_{k_i}}$

$$\Rightarrow \mathcal{E}_{k_1} \mathcal{E}_{k_2} |\vec{v}_1 - \vec{v}_2| = \mathcal{E}_{k_1} \mathcal{E}_{k_2} \left| \frac{\vec{k}_1}{\mathcal{E}_{k_1}} - \frac{\vec{k}_2}{\mathcal{E}_{k_2}} \right| = \left| \mathcal{E}_{k_2} \vec{k}_1 - \mathcal{E}_{k_1} \vec{k}_2 \right|$$

$$= \sqrt{(k_1\cdot k_2)^2 - m_1^2 m_2^2} \Rightarrow \text{Lorentz-invariant.}$$

$$\Big(k_1\cdot k_2 = \mathcal{E}_1 \mathcal{E}_2 - \vec{k}_1\cdot\vec{k}_2 \Rightarrow \sqrt{(k_1\cdot k_2)^2 - m_1^2 m_2^2} = \Big((\mathcal{E}_1\mathcal{E}_2 - \vec{k}_1\cdot\vec{k}_2)^2 - m_1^2 m_2^2 \Big)^{1/2} = \Big(\mathcal{E}_1^2 \mathcal{E}_2^2 + (\vec{k}_1\cdot\vec{k}_2)^2 - 2\mathcal{E}_1\mathcal{E}_2 \vec{k}_1\cdot\vec{k}_2 - m_1^2 m_2^2 \Big)^{1/2}$$

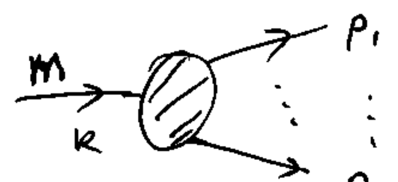
$$= \underset{\vec{k}_1 \parallel \vec{k}_2}{(\text{collinear case})} = \Big(\mathcal{E}_1^2\mathcal{E}_2^2 + (\mathcal{E}_1^2 - m_1^2)(\mathcal{E}_2^2 - m_2^2) - 2\mathcal{E}_1\mathcal{E}_2 \vec{k}_1\cdot\vec{k}_2 - m_1^2 m_2^2 \Big)^{1/2} = \Big(2\mathcal{E}_1^2\mathcal{E}_2^2 - m_1^2\mathcal{E}_2^2 - m_2^2\mathcal{E}_1^2 - 2\mathcal{E}_1\mathcal{E}_2 \vec{k}_1\cdot\vec{k}_2 \Big)^{1/2} = \Big(\mathcal{E}_2^2 \vec{k}_1^{\,2} + \mathcal{E}_1^2 \vec{k}_2^{\,2} - 2\mathcal{E}_1\mathcal{E}_2 \vec{k}_1\cdot\vec{k}_2 \Big)^{1/2}$$

$$= \left| \mathcal{E}_2 \vec{k}_1 - \mathcal{E}_1 \vec{k}_2 \right|$$

$\Rightarrow \mathcal{E}_{k_1}\mathcal{E}_{k_2} |\vec{v}_1 - \vec{v}_2|$ is Lorentz invariant iff $\vec{v}_1, \vec{v}_2, \mathcal{E}_{k_1}, \mathcal{E}_{k_2}$ are taken in a frame where $\vec{v}_1 \parallel \vec{v}_2$!

<u>Decay rate</u>: imagine one particle decaying into N particles:

work in particle's rest frame



$\Rightarrow$ start from the x-section:

$$\frac{1}{2\mathcal{E}_{k_1} 2\mathcal{E}_{k_2} |\vec{v}_1 - \vec{v}_2|} \longrightarrow \frac{1}{2M}, \quad m \sim \text{particle mass}$$

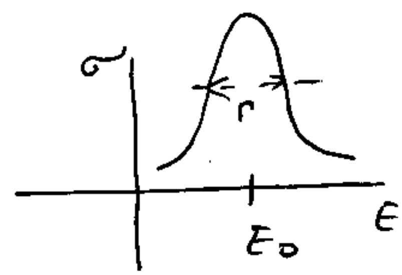
$$d\Gamma = \frac{1}{2m} \prod_{i=1}^{N} \frac{d^3 p_i}{(2\pi)^3 2E_i} \, |M(m; p_1, \ldots, p_N)|^2 (2\pi)^4 \delta\left(k - \sum_{j=1}^{N} p_j\right)$$

~ decay rate

$$\Gamma = \frac{\#\text{ decays per unit time}}{\#\text{ of particles}}$$

Breit-Wigner formula for scattering amplitude:

$$f(E) \sim \frac{1}{E - E_0 + i\,\Gamma/2}$$



$$\Rightarrow \sigma(E) \propto |f(E)|^2 \sim \frac{1}{(E-E_0)^2 + \frac{\Gamma^2}{4}}$$

width of resonance peak!

$\Rightarrow$ NB in calculating $|M|^2$ sum over spins, etc in the final state, average over them in the initial state.

Decay of the Z-boson

$\Rightarrow$ let's calculate decay rate of the Z-boson

$\Rightarrow$ the Z-boson interaction Lagrangian is:

$$\mathcal{L}_Z = \frac{g}{4\cos\theta_W} \Big\{ \bar{\nu}_e \gamma \cdot Z (1-\gamma_5) \nu_e + 2\sin^2\theta_W \, \bar{e} \gamma \cdot Z (1+\gamma_5) e \, +$$

$$+ (2\sin^2\theta_W - 1) \, \bar{e} \gamma \cdot Z (1-\gamma_5) e \underbrace{+}_{+(\mu,\tau)} \bar{U} \gamma \cdot Z \Big[(1-\gamma_5)\left(1 - \tfrac{4}{3}\sin^2\theta_W\right) - (1+\gamma_5)$$

$$\cdot \tfrac{4}{3} \sin^2\theta_W \Big] U - \bar{D} \gamma \cdot Z \Big[(1-\gamma_5)\left(1 - \tfrac{2}{3}\sin^2\theta_W\right) - (1+\gamma_5) \tfrac{2}{3} \sin^2\theta_W \Big] D \Big\}$$

$Z \to W^+W^-$, $Z \to \gamma Z$, $Z \to W^+W^- Z, \ldots$ are all prohibited by energy conservation! $M_Z \simeq 91\,\text{GeV} < 2M_W = 2 \cdot 80\,\text{GeV} \ldots$