Faddeev - Popov Ghost

We want to quantize a gauge theory:

$$\mathcal{L} = -\frac{1}{4} F^a_{\mu\nu} F^{a\mu\nu}$$ (consider a general non-Abelian case).

The generating functional is

$$Z = \int \mathcal{D} A_\mu \, e^{iS} = \int \mathcal{D} A_\mu \, e^{i\int d^4x \left(-\frac{1}{4}\right) F^a_{\mu\nu} F^{a\mu\nu}} =$$

$$= \int \mathcal{D} \bar{A}_\mu \, e^{iS} \cdot \int \mathcal{D} \Lambda$$

where $\bar{A}_\mu$ is the field in one particular gauge, Λ is the gauge transformation.

Problem: $\int \mathcal{D}\Lambda = \infty \Rightarrow Z = \infty \Rightarrow$ bad!

Even worse is the need to pick a gauge: consider abelian field A_μ: $\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} = -\frac{1}{4}(\partial_\mu A_\nu - \partial_\nu A_\mu)(\partial^\mu A^\nu - \partial^\nu A^\mu) = \frac{1}{2} A^\mu \cdot \underbrace{[g_{\mu\nu}\partial^2 - \partial_\mu \partial_\nu]}_{(D^{-1})_{\mu\nu}} A^\nu$

$\Rightarrow$ to find photon propagator need to solve

$$[g_{\mu\nu}\partial^2 - \partial_\mu\partial_\nu] D^{\nu\rho}(x) = \delta_\mu^\rho \, \delta^4(x).$$

$\Rightarrow$ act with $\partial^\mu \Rightarrow (\partial_\nu \partial^2 - \partial^2 \partial_\nu) D^{\nu\rho} = 0 = \partial^\rho \delta^4(x)$

$\Rightarrow$ this can not be true $\Rightarrow$ the operator has no inverse! $\Rightarrow$ no photon propagator?

However, if we choose a gauge, e.g. $\partial_\mu A_\mu = 0$

$\Rightarrow \quad \mathcal{L} = \frac{1}{2} A^\mu \Box A^\nu \Rightarrow g_{\mu\nu} \Box D^{\nu\rho}(x) = \delta_\mu{}^\rho \delta^4(x).$

$\Rightarrow$ easy to invert!

$\Rightarrow$ Need to fix the gauge!

Start with $\quad Z = \int \mathcal{D}A_\mu \, e^{iS}$.

Insert into the integrand

$$1 = \int \mathcal{D}\Lambda \; \delta(\Lambda) = \int \mathcal{D}\Lambda \cdot \delta(G(A^\Lambda)) \det\left(\frac{\delta G(A^\Lambda)}{\delta \Lambda}\right)$$

where $G(A) = 0$ is the gauge condition we want to impose, e.g. $G(A) = \partial_\mu A^\mu$ for covariant gauge. Now

$$Z = \int \mathcal{D}A_\mu \, e^{iS(A_\mu)} \int \mathcal{D}\Lambda \; \delta(G(A^\Lambda)) \det\left(\frac{\delta G(A^\Lambda)}{\delta \Lambda}\right)$$

Change the order of integration & define a new field $A'_\mu = A_\mu^\Lambda$ to write (dropping the prime)

$$Z = \underbrace{\int \mathcal{D}\Lambda}_{\text{still } \infty \text{, but an overall factor}} \cdot \int \mathcal{D}A_\mu \, e^{iS(A_\mu)} \, \delta(G(A_\mu)) \det\left(\frac{\delta G(A^\Lambda)}{\delta \Lambda}\right).$$ A3

still ∞, but
an overall factor $\Rightarrow$ cancels in correlators like

$$\langle A_\mu(x) A_\nu(y) \rangle = \frac{1}{Z} \cdot \int \mathcal{D}A_\mu \; A_\mu(x) A_\nu(y) \, e^{iS}.$$

A trick: choose $G(A) = \partial_\mu A^\mu - \omega(x) \Rightarrow$

$$\Rightarrow \delta(G(A)) = \delta(\overbrace{\partial_\mu A^\mu}^{\bar{G}(A)} - \omega(x))$$

Nothing else in Z depends on $\omega(x) \Rightarrow$ integrate over $\omega(x)$:

$$1 = \underbrace{N(\xi)}_{\text{Norm}} \int \mathcal{D}\omega(x) \, e^{-i\int d^4x \frac{\omega^2}{2\xi}}$$

$$\Rightarrow Z = \int \mathcal{D}\Lambda \cdot N(\xi) \int \mathcal{D}A_\mu \, e^{iS(A_\mu)} \int \mathcal{D}\omega \, e^{-i\int d^4x \frac{\omega^2}{2\xi}}$$

$$\cdot \, \delta(\bar{G}(A) - \omega(x)) \det\left(\frac{\delta G(A^\Lambda)}{\delta\Lambda}\right) = \int \mathcal{D}\Lambda \cdot N(\xi) \cdot$$

$$\cdot \int \mathcal{D}A_\mu \cdot \det\left(\frac{\delta G(A^\Lambda)}{\delta \Lambda}\right) \cdot e^{iS(A_\mu)} \cdot e^{-i\int d^4x \frac{1}{2\xi}(G(A))^2}.$$

$\int \mathcal{D}\Lambda \cdot N(\xi)$ is an unimportant overall factor.

What do we do with $\det\left(\frac{\delta G}{\delta\Lambda}\right)$?

We have

$$Z \sim \int \mathcal{D} A_\mu \; \det\left(\frac{\delta G(A^\Lambda)}{\delta \Lambda}\right) \cdot e^{\,i S(A) - i \int d^4x \, \frac{[G(A)]^2}{2\xi}}$$

We want to put det into the exponent ~ make it a part of the Lagrangian.

Note that $\displaystyle\int_{-\infty}^{\infty} dx \; e^{-a x^2} = \sqrt{\frac{\pi}{a}} \quad \Rightarrow$

$$\Rightarrow \int_{-\infty}^{\infty} dx_1 \ldots dx_n \; e^{-a_1 x_1^2 - \ldots - a_n x_n^2} = (\pi)^{n/2} \cdot \frac{1}{\sqrt{a_1 a_2 \ldots a_n}}$$

$$= \frac{\pi^{n/2}}{\sqrt{\det A}}, \qquad A = \begin{pmatrix} a_1 & & 0 \\ & a_2 & \\ 0 & & \ddots \; a_n \end{pmatrix} \text{ a diagonal matrix.}$$

Similarly, for $\forall A$ get $\boxed{\displaystyle\int_{-\infty}^{\infty} d^n x \; e^{-x^T A x} = \frac{\pi^{n/2}}{\sqrt{\det A}}}$

$\Rightarrow$ can absorb $\dfrac{1}{\sqrt{\det A}}$ into exponent. But here we have $\det A$!

Grassmann quantities: η is a Grassmann #

$\Rightarrow$ if η is single-component $\Rightarrow f(\eta) = A + B\eta$

$(\eta^2 = 0, \; \eta^3 = 0, \ldots) \Rightarrow \dfrac{df}{d\eta} = B \Rightarrow \dfrac{d^2 f}{d\eta^2} = 0$

$\Rightarrow$ no inverse to differentiation?

To define integrals note:

$$\int d\eta\, f(\eta) = \int d\eta\,(A+B\eta) = \Big|\, \eta \to \eta + \theta \;= \int d\eta\,(A+B\eta+B\theta)$$

$$\Rightarrow \int d\eta \cdot A = \int d\eta\,(A+B\theta) \Rightarrow \int d\eta = 0$$

$$\int d\eta\, B\eta = B \quad \text{(linear in } B\text{, adjust constant to } 1\text{)}$$

$$\Rightarrow \int d\eta \cdot \eta = 1$$

Complex η: $\bar{\eta}$ is c.c. $\int d\eta = \int d\bar{\eta} = 0$, $\int d\eta \cdot \eta = \int d\bar{\eta}\,\bar{\eta} = 1$.

$$\int d\bar{\eta}\, d\eta\, e^{-b\bar{\eta}\eta} = \int d\bar{\eta} \int d\eta\,(1-b\bar{\eta}\eta) = \int d\bar{\eta}\cdot(+)b\bar{\eta} = 1b = b.$$

Two-component Grassmann #'s: $\eta = \begin{pmatrix}\eta_1\\ \eta_2\end{pmatrix}$, $\bar{\eta} = \begin{pmatrix}\bar{\eta}_1\\ \bar{\eta}_2\end{pmatrix}$

$$\Rightarrow \bar{\eta}\eta = \bar{\eta}_1\eta_1 + \bar{\eta}_2\eta_2$$

$$(\bar{\eta}\eta)^2 = (\bar{\eta}_1\eta_1 + \bar{\eta}_2\eta_2)^2 = 2\bar{\eta}_1\eta_1\bar{\eta}_2\eta_2$$

$\int d\bar{\eta}\, d\eta\, e^{-\bar{\eta}^T M \eta}$ need to find, M is a 2×2 matrix

$$\Rightarrow e^{-\bar{\eta}^T M \eta} = e^{-(\bar{\eta}_1\ \bar{\eta}_2)\begin{pmatrix}M_{11} & M_{12}\\ M_{21} & M_{22}\end{pmatrix}\begin{pmatrix}\eta_1\\ \eta_2\end{pmatrix}} =$$

$$= \exp\left\{-(\bar{\eta}_1\ \bar{\eta}_2)\begin{pmatrix}M_{11}\eta_1 + M_{12}\eta_2\\ M_{21}\eta_1 + M_{22}\eta_2\end{pmatrix}\right\} = \exp\Big\{-\big[\bar{\eta}_1(M_{11}\eta_1 + M_{12}\eta_2) +$$

$$+\bar\eta_2\left(M_{21}\eta_1+M_{22}\eta_2\right)\Big]\Big\} = 1-\bar\eta_1(M_{11}\eta_1+M_{12}\eta_2)$$

$$-\bar\eta_2(M_{21}\eta_1+M_{22}\eta_2)+\frac{1}{2}\,2\,\bar\eta_1\,(M_{11}\eta_1+M_{12}\eta_2)\,\bar\eta_2\,(M_{21}\eta_1+$$

$$+M_{22}\eta_2) = 1-\bar\eta_1(M_{11}\eta_1+M_{12}\eta_2)-\bar\eta_2(M_{21}\eta_1+M_{22}\eta_2)$$

$$+\bar\eta_1\eta_1\,\bar\eta_2\eta_2\,\left(M_{11}M_{22}-M_{12}M_{21}\right)$$

$$\Rightarrow \int d\bar\eta\, d\eta\; e^{-\bar\eta^T M\eta} = \int \overbrace{d\bar\eta_1\, d\bar\eta_2\, d\eta_2\, d\eta_1}^{d\bar\eta_1 d\eta_1\, d\bar\eta_2 d\eta_2}\;\bar\eta_1\eta_1\bar\eta_2\eta_2\cdot$$

$$\cdot\left(M_{11}M_{22}-M_{12}M_{21}\right) = \det M.$$

$\Rightarrow$ can show for $\forall$ dimension

$$\int d\bar\eta\, d\eta\; e^{-\bar\eta^T M\eta} = \det M$$

$$\Rightarrow \det\left[\frac{\delta G(A^\Lambda)}{\delta\Lambda}\right] = \int \mathcal{D}\eta\,\mathcal{D}\bar\eta\; e^{-i\int d^4x\,\bar\eta\,\frac{\delta G}{\delta\Lambda}\eta}.$$

$$\Rightarrow Z \sim \int \underbrace{\mathcal{D}A_\mu}_{}{}^{\mathcal{D}\eta\mathcal{D}\bar\eta}\; e^{iS(A)-i\int d^4x\frac{[G(A)]^2}{2\xi}-i\int d^4x\,\bar\eta\,\frac{\delta G}{\delta\Lambda}\eta}$$

η ~ Faddeev-Popov ghost.

<u>Covariant gauge</u>: $G(A)=\partial_\mu A^{a\mu}$ $\Rightarrow$ gauge

transform is $A_\mu \to \Lambda A_\mu \Lambda^{-1}-\frac{i}{g}(\partial_\mu\Lambda)\Lambda^{-1}$

$\Rightarrow$ for infinitesimal gauge transform:

$$\Lambda = 1 + i\,\alpha^a T^a \Rightarrow A_\mu^a T^a \to (1 + i\,\alpha^a T^a) A_\mu \cdot$$

$$\cdot (1 - i\,\alpha^b T^b) - \frac{i}{g}\, i T^a (\partial_\mu \alpha^a)(1 - i\,\alpha^b T^b) =$$

$$= A_\mu + i\,[\alpha, A_\mu] + \frac{1}{g}\,\partial_\mu \alpha = T^a A_\mu'^a$$

$$\Rightarrow A_\mu'^a = A_\mu^a + i \cdot i f^{abc} \alpha^b A_\mu^c + \frac{1}{g}\,\partial_\mu \alpha^a$$

$$= A_\mu^a + f^{abc} A_\mu^b \alpha^c + \frac{1}{g}\,\partial_\mu \alpha^a =$$

$$= A_\mu^a + \frac{1}{g}\, \mathcal{D}_\mu \alpha^a$$

$$\Rightarrow \frac{\delta G}{\delta \Lambda} = \frac{\delta G}{\delta \alpha} = \frac{\delta (\partial_\mu A^{a\mu})}{\delta \alpha} = \partial_\mu \frac{1}{g} \mathcal{D}^\mu$$

$$\Rightarrow \mathcal{L} = -\frac{1}{4} F_{\mu\nu}^a F^{a\mu\nu} - \frac{1}{2\xi} (\partial_\mu A^{a\mu})^2 - \bar{\eta}\, \partial_\mu \mathcal{D}^\mu \eta$$

$$\Rightarrow \mathcal{L} = -\frac{1}{4} F_{\mu\nu}^a F^{a\mu\nu} - \frac{1}{2\xi} (\partial_\mu A^{a\mu})^2 + (\partial_\mu \bar{\eta}) \mathcal{D}^\mu \eta$$

as advertised.

<u>Light-cone gauge:</u> $G(A) = n \cdot A^a = n_\mu A^{a\mu} \Rightarrow$

$$\Rightarrow \frac{\delta G}{\delta \alpha} = n_\mu \frac{\delta A^{a\mu}}{\delta \alpha} = n_\mu \frac{1}{g} \mathcal{D}^\mu = \frac{1}{g}\, n_\mu \big(\partial^\mu - ig\underbrace{[A^\mu, \dots]}_{0}\big)$$

as $n \cdot A = 0$

$= \frac{1}{g} n \cdot \partial$ $\Rightarrow$ is A_μ - independent $\Rightarrow$

$\Rightarrow$ ~~again~~ gives only an overall factor in Z

$\Rightarrow$ do not need it, no need to introduce the ghost!