Homework Set No. 1, Physics 880.02

Deadline – Thursday, January 22, 2009

1. Consider a free (real) scalar theory with the Lagrangian density

$$\mathcal{L} = \frac{1}{2}\partial_\mu\varphi\partial^\mu\varphi - \frac{m^2}{2}\varphi^2.$$

Define the Hamiltonian by

$$H(t) = \int d^3x \left[\pi(\vec{x},t)\,\dot{\varphi}(\vec{x},t) - \mathcal{L}\right].$$

a. (3 pts) Show that for classical field configurations

$$\frac{d}{dt}H(t) = 0.$$

b. (2 pts) Write $H(t)$ in terms of π and φ with no $\dot{\varphi}$'s appearing.

c. (5 pts) Now imagine that the field is quantized. Use canonical quantization commutators

$$[\varphi(\vec{x},t), \pi(\vec{x}\,',t)] = i\delta(\vec{x} - \vec{x}\,')$$

(with all other commutators being zero) to show that $H(t)$ (now an operator) generates time translations, i.e., show that

$$\begin{aligned} i\,\partial_0\varphi &= [\varphi, H(t)] \\ i\,\partial_0\pi &= [\pi, H(t)]\,. \end{aligned}$$

2. The same as in problem 1, but now for Dirac field: start with a theory with Lagrangian density

$$\mathcal{L} = \bar{\psi}(i\gamma^\mu\partial_\mu - m)\psi.$$

a. (3 pts) Construct a Hamiltonian $H(t)$ and show that for classical field configurations

$$\frac{d}{dt}H(t) = 0.$$

b. (2 pts) Write $H(t)$ in terms of π and ψ.

c. (5 pts) For quantized field ψ use the anti-commutation relations

$$\left\{\psi_\alpha(\vec{x},t), \psi^\dagger_\beta(\vec{x}\,',t)\right\} = \delta_{\alpha\beta}\,\delta(\vec{x} - \vec{x}\,')$$

to show that

$$\begin{aligned} i\,\partial_0\psi_\alpha &= [\psi_\alpha, H(t)] \\ i\,\partial_0\bar{\psi}_\alpha &= \left[\bar{\psi}_\alpha, H(t)\right]. \end{aligned}$$

3. Consider a massive Dirac field ψ with mass m.

a. (2 pts) Starting with Dirac equation

$$[i\gamma^\mu \partial_\mu - m]\psi = 0$$

derive an equation for $\bar{\psi}$.

b. (3 pts) Using the result of part **a** show that the electromagnetic current

$$j_\mu = \bar{\psi}\gamma_\mu\psi$$

is conserved at the classical level, i.e., show that $\partial_\mu j^\mu = 0$.

c. (2 pts) Now consider an interaction of the Dirac field ψ with some gauge (photon) vector field A_μ. The corresponding term in the Lagrangian is

$$\mathcal{L}_{\text{int}} = e\,\bar{\psi}\gamma^\mu A_\mu \psi$$

Using the result of part **b** show that $\int d^4x \mathcal{L}_{\text{int}}$ is invariant under the gauge transformation

$$A_\mu \to A'_\mu = A_\mu + \partial_\mu \Lambda.$$

d. (3 pts) Defining $\gamma_5 = i\gamma_0\,\gamma_1\,\gamma_2\,\gamma_3$ use the anti-commutation relations for γ-matrices $\{\gamma_\mu, \gamma_\nu\} = 2g_{\mu\nu}$ to show that

$$\{\gamma_\mu, \gamma_5\} = 0.$$

Use this result to show that the axial vector current

$$j_{5\,\mu} = \bar{\psi}\gamma_\mu\gamma_5\psi$$

is conserved at the classical level but only for *massless* Dirac fields: $\partial_\mu j_5^\mu = 0$ (you may put $m = 0$ from the start in this part). (Note however that we can not couple the axial current to the gauge field: there is no terms like $A^\mu j_{5\,\mu}$ in the Lagrangian as they are parity-odd.)