Homework Set No. 1, Physics 880.02

Deadline – Tuesday, April 28, 2009

1. (10 pts) Assuming for simplicity that there are only two neutrino families (electron and muon neutrinos), estimate the mass squared difference Δm^2 between the two neutrino mass states given that the flux of electron neutrinos as they hit Earth is only about 1/3 of the electron neutrino flux as they leave the Sun (that is 2/3 of electron neutrinos oscillate into muon ones). Assume that the distance between the Earth and the Sun is 1.5×10^{11} m, neutrinos have the energy of about 1 MeV and the mixing angle θ is $45°$.

2. In class we have discussed the renormalization group (Callan-Symanzik) equation

$$\left[\mu^2 \frac{\partial}{\partial \mu^2} + \beta(\alpha_\mu)\, \frac{\partial}{\partial \alpha_\mu}\right] \mathcal{M}(Q^2/\mu^2, \alpha_\mu) \;=\; 0$$

for a dimensionless observable $\mathcal{M}$ in zero-mass QCD. Suppose the beta-function is exactly

$$\beta(\alpha_\mu) \;=\; -\beta_2\, \alpha_\mu^2 - \beta_3\, \alpha_\mu^3$$

in some renormalization scheme.

a. (5 pts) Show that

$$\frac{1}{\beta(\alpha)} \;=\; -\frac{1}{\beta_2\, \alpha^2} + \frac{\beta_3}{{\beta_2}^2\, \alpha} + \text{analytic terms in}\, \alpha.$$

b. (10 pts) Define the Λ-parameter by

$$\Lambda^2 \;=\; \mu^2\, c\, \exp\left(-\int\limits_{\alpha_0}^{\alpha_\mu} \frac{d\alpha}{\beta(\alpha)}\right)$$

with α_0 and c some constants. Show that one can choose α_0 and c such that

$$\ln \frac{\mu^2}{\Lambda^2} \;=\; \frac{1}{\beta_2\, \alpha_\mu} + \frac{\beta_3}{{\beta_2}^2}\, \ln(\beta_2\, \alpha_\mu) + o(\alpha_\mu). \qquad (1)$$

c. (10 pts) Solve Eq. (1) above for the running coupling $\alpha_\mu \equiv \alpha(\mu^2)$ in terms of $\ln(\mu^2/\Lambda^2)$ assuming that $\ln(\mu^2/\Lambda^2) \gg 1$ and keeping all terms which decrease with increasing μ^2 less rapidly than $\ln^{-3}(\mu^2/\Lambda^2)$.

d. (10 pts) Show that

$$\Lambda^2 \;=\; \mu^2\, e^{-\frac{1}{\beta_2\, \alpha_\mu}}\, (\beta_2\, \alpha_\mu)^{-\beta_3/\beta_2^2}\, [1 + o(\alpha_\mu)] \qquad (2)$$

and prove that this Λ^2 is μ-independent.