Homework Set No. 3, Physics 880.02

Deadline – Tuesday, February 17, 2009

1. Just like in class consider 2-flavor QCD with massless quarks:

$$q(x) = \begin{pmatrix} u(x) \\ d(x) \end{pmatrix}, \quad q_{L,R} = \frac{1 \mp \gamma_5}{2}\, q.$$

The Lagrangian is

$$\mathcal{L} \;=\; \bar{q}_L i\gamma \cdot \partial q_L + \bar{q}_R i\gamma \cdot \partial q_R.$$

The left- and right-handed isospin currents are

$$j_L^{i\,\mu} = \bar{q}_L \gamma^\mu \frac{\sigma^i}{2} q_L \quad \text{and} \quad j_R^{i\,\mu} = \bar{q}_R \gamma^\mu \frac{\sigma^i}{2} q_R$$

with the charges

$$Q_L^i(t) = \int d^3x \, j^i_{L,0}(\vec{x}, t) \quad \text{and} \quad Q_R^i(t) = \int d^3x \, j^i_{R,0}(\vec{x}, t).$$

a. (10 pts) Using anti-commutation relations

$$\left\{ q^a_\alpha(\vec{x}, t), q^{\dagger\, b}_\beta(\vec{x}', t) \right\} = \delta^{ab}\, \delta_{\alpha\beta}\, \delta(\vec{x} - \vec{x}')$$

show that for any matrices Γ_1 and Γ_2 (which are matrices both in Dirac and flavor spaces) the following relation holds

$$\left[q^\dagger(\vec{x}', t)\, \Gamma_1\, q(\vec{x}', t),\; q^\dagger(\vec{x}, t)\, \Gamma_2\, q(\vec{x}, t) \right] \;=\; \delta(\vec{x} - \vec{x}')\, q^\dagger(\vec{x}, t)\, [\Gamma_1, \Gamma_2]\, q(\vec{x}, t).$$

b. (10 pts) Using the result of part **a** show that Q_L^i and Q_R^i form a chiral algebra of $SU(2)_L \otimes SU(2)_R$, i.e., prove that

$$\begin{aligned} \left[Q_L^i, Q_L^j \right] \;&=\; i\, \epsilon_{ijk}\, Q_L^k \\ \left[Q_R^i, Q_R^j \right] \;&=\; i\, \epsilon_{ijk}\, Q_R^k \\ \left[Q_L^i, Q_R^j \right] \;&=\; 0. \end{aligned}$$

c. (5 pts) Now add a mass term to the Lagrangian:

$$\mathcal{L} \;=\; \bar{q} i\gamma \cdot \partial q - m\, \bar{q} q \quad \text{with} \quad m = \begin{pmatrix} m_u & 0 \\ 0 & m_d \end{pmatrix}.$$

Find the divergence of the axial vector isospin current $\partial_\mu j_5^{i\,\mu}$ where $j_5^{i\,\mu} = j_R^{i\,\mu} - j_L^{i\,\mu}$. Is the chiral $SU(2)_L \otimes SU(2)_R$ symmetry still a symmetry of the Lagrangian with massive quarks? Does the mass term affect the $SU(2)_L \otimes SU(2)_R$ chiral algebra that you derived in part **b**?

2. (10 pts) Consider the Lagrangian

$$\mathcal{L} = \partial_\alpha \phi^* \partial^\alpha \phi + \mu^2 \, \phi^* \, \phi - \frac{\lambda}{2} \, (\phi^* \, \phi)^2$$

with ϕ a complex scalar field and $\mu^2 > 0$. Write

$$\phi = \frac{\chi_1 + i \, \eta}{\sqrt{2}}$$

with χ_1 and η real fields and with $\chi_1 = v + \chi$, where v is a real number and χ is a real dynamical variable (a field). What is the value of v for which the linear terms in χ disappear from the Lagrangian? Write the Lagrangian in terms of χ and η and give masses of the particles in the theory.

3. Consider σ-model without the nucleons (quarks). The Lagrangian is

$$\mathcal{L} = \frac{1}{2} \left[\partial_\mu \sigma \, \partial^\mu \sigma + \partial_\mu \vec{\pi} \, \partial^\mu \vec{\pi} \right] + \frac{\mu^2}{2} \left[\sigma^2 + \vec{\pi}^2 \right] - \frac{\lambda}{4} \left[\sigma^2 + \vec{\pi}^2 \right]^2 \tag{1}$$

where as usual σ is a scalar field and $\vec{\pi} = (\pi^1, \pi^2, \pi^3)$ is the three-component scalar (pion) field.

a. (10 pts) Unlike the discussion in class choose the vacuum configuration to be

$$\langle \pi^1 \rangle = \langle \pi^2 \rangle = \langle \sigma \rangle = 0, \quad \langle \pi^3 \rangle = v. \tag{2}$$

Find the value of v minimizing the potential in the σ-model Lagrangian (1). Writing

$$\pi^3 = v + \pi'^3$$

express the Lagrangian (1) in terms of fields π^1, π^2, σ and π'^3 and find masses of the particles in the theory.

b. (5 pts) Find the symmetry group corresponding to the non-Abelian symmetry still left in the Lagrangian expressed in terms of fields π^1, π^2, σ and π'^3 that you found in part **a**? In other words, the original $SU(2)_L \otimes SU(2)_R$ chiral symmetry of the Lagrangian (1) is spontaneously broken down by the choice of vacuum in (2). What symmetry is it broken down to?