Homework Set No. 4, Physics 880.02
Deadline – Thursday, February 26, 2009

1. (10 pts) Consider a non-Abelian gauge theory with the gauge field A_μ^a and the Lagrangian

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu}^a F^{a\,\mu\nu}.$$

Here

$$F_{\mu\nu}^a = \partial_\mu A_\nu^a - \partial_\nu A_\mu^a + g\, f^{abc} A_\mu^b A_\nu^c$$

with f^{abc} the structure constants of the gauge group $SU(N)$.

Write the equations of motion for this theory. If we define $J^{a\,\mu}$ by

$$\partial_\nu F^{a\,\nu\mu} = J^{a\,\mu}$$

what is $J^{a\,\mu}$ for the above Lagrangian?

2. (10 pts) Consider the Lagrangian for a complex scalar field ϕ coupled to an Abelian gauge field A_μ:

$$\mathcal{L} = (\partial_\mu + i\,g\,A_\mu)\phi^* (\partial_\mu - i\,g\,A_\mu)\phi + \mu^2 \phi^* \phi - \lambda\,(\phi^* \phi)^2 - \frac{1}{4} F_{\mu\nu} F^{\mu\nu}$$

with $\mu^2, \Lambda > 0$ positive constants. Like in problem 2 of HW3 write

$$\phi = \frac{\chi_1 + i\,\eta}{\sqrt{2}}$$

with χ_1 and η real fields and with $\chi_1 = v + \chi$, where v is a real number and χ is a real field. What is the value of v for which the linear terms in χ disappear from the Lagrangian? Write the Lagrangian in terms of A_μ, χ and η. What are the *physical* particles and what are their masses?

3. Consider the Higgs potential in the Standard Model:

$$V(\phi) = -\mu^2 \phi^\dagger \phi + \lambda\,(\phi^\dagger \phi)^2 \tag{1}$$

with the Higgs field

$$\phi = \begin{pmatrix} \phi^+ \\ \phi^0 \end{pmatrix} = \begin{pmatrix} \phi^1 \\ \phi^2 \end{pmatrix}.$$

The potential is $SU(2) \otimes U(1)$ invariant.

a. (5 pts) In principle one can write $SU(2) \otimes U(1)$ quadric invariant of the form

$$V_1(\phi) \;=\; \lambda_1 \, (\phi^\dagger \, \vec{\tau} \, \phi) \,\cdot\, (\phi^\dagger \, \vec{\tau} \, \phi) \tag{2}$$

with $\vec{\tau}$ the Pauli matrices and λ_1 a constant. Show that this quadric term can be reduced to that in $V(\phi)$ using the Fierz identity

$$\sum_{a=1}^{3} \left(\frac{\tau^a}{2}\right)_{ij} \left(\frac{\tau^a}{2}\right)_{kl} \;=\; \frac{1}{2}\,\delta_{jk}\,\delta_{il} - \frac{1}{4}\,\delta_{ij}\,\delta_{kl}.$$

b. (5 pts) The same as in part **a** for

$$V_2(\phi) \;=\; \lambda_1 \sum_{a,b=1}^{3} (\phi^\dagger \, \tau^a \, \tau^b \, \phi) \, (\phi^\dagger \, \tau^a \, \tau^b \, \phi). \tag{3}$$