Last time: talked more about chiral symmetry

$N_f = 2$ $\quad \boxed{\mathcal{L} = \bar{q}_L \, i\gamma\cdot\partial q_L + \bar{q}_R \, i\gamma\cdot\partial q_R}$

$\Rightarrow$ $SU(2)_L \otimes SU(2)_R$ chirally symmetric.

Conserved currents:

$$\boxed{\begin{aligned} j_L^{i\mu} &= \bar{q}_L \gamma^\mu \frac{\sigma^i}{2} q_L \\ j_R^{i\mu} &= \bar{q}_R \gamma^\mu \frac{\sigma^i}{2} q_R \end{aligned}}$$

Conserved charges:

$$\boxed{Q_{L,R}^i(t) = \int d^3x \, j_{L,R}^{i\,0}(\vec{x}, t)}$$

$$\frac{dQ_{L,R}^i}{dt} = 0$$

The charges are generators of $SU(2)_L \otimes SU(2)_R$:

$$\boxed{\begin{aligned} [Q_L^i, Q_L^j] &= i\varepsilon^{ijk} Q_L^k \\ [Q_R^i, Q_R^j] &= i\varepsilon^{ijk} Q_R^k \\ [Q_L^i, Q_R^j] &= 0 \end{aligned}}$$

Problem:
$(N_f = 3)$

$$\boxed{SU(3)_L \otimes SU(3)_R}$$

$$\Big\downarrow \; m_u = m_d = m_s \neq 0$$

$\Rightarrow$ need SSB here (spontaneous symmetry breaking)

$$\boxed{SU(3)}$$

$$\Big\downarrow \; m_u \neq m_d \neq m_s \neq 0$$

Nothing

Both symmetries are equally broken, but $SU(3)$ is still observed in nature, while $SU(3)_L \otimes SU(3)_R$ is not!

Def. SSB: a symmetry manifest in $\mathcal{L}$, H, but not respected by ground state.

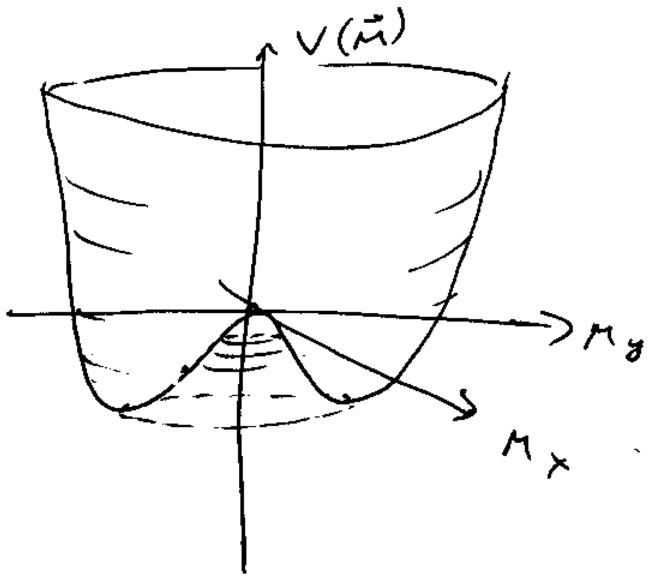
<u>Example</u>: Ising model, $\begin{matrix}\uparrow & \uparrow & \uparrow \\ \uparrow & \uparrow & \uparrow\end{matrix}$ vs $\begin{matrix}\downarrow & \downarrow & \downarrow \\ \downarrow & \downarrow & \downarrow\end{matrix}$.

<u>Landau - Ginzburg theory of ferromagnetism</u>

(cont'd)

$$H = \int d^3x \Big[(\nabla_i M_j)^2 + \underbrace{\mu^2 (T - T_c)\vec{M}^2 + \lambda (\vec{M}^2)^2}_{V} \Big]$$

$\lambda, \mu^2 > 0$, $T \sim$ temperature



for $T < T_c$

$M^i \to M'^i = R^{ij} M^j$

SO(3) rotational symmetry

Ground state:

$$|\vec{M}_{gr.}| = \sqrt{\frac{\mu^2 (T_c - T)}{2\lambda}} \neq 0$$

Direction of $\vec{M}_{vac}$ is random $\sim$ SSB!

$x'_i = R_{ij} x_j \Rightarrow |\vec{x}'|^2 = |\vec{x}|^2 \Rightarrow R_{ij} x_j R_{ik} x_k = x_i x_i$

$\Rightarrow R_{ij} R_{ik} = \delta_{jk} \Rightarrow R \cdot R^T = R^T R = \mathbb{1} \Rightarrow$ forget reflections $\Rightarrow$ require $\det R = +1 \Rightarrow SO(3)$

~ a group of special ($\det = +1$) real orthogonal ($RR^T = R^T R = \mathbb{1}$) 3×3 matrices.

$\Rightarrow$ for $T < T_c$ the ground state is at the minima

$\Rightarrow \quad \mu^2 (T - T_c)\, 2|\vec{M}| + 4\lambda |\vec{M}|^3 = 0$

$$\Rightarrow \quad |\vec{M}_{vac}| = \sqrt{\frac{\mu^2 (T_c - T)}{2\lambda}}$$

$\Rightarrow$ however, direction of $\vec{M}$ is chosen <u>spontaneously</u>!

Say, $M_{vac} = \sqrt{\frac{\mu^2 (T_c - T)}{2\lambda}}\, \hat{x} \equiv |0\rangle$

define generators of $SO(3)$: $L_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}$, $L_2 = \begin{pmatrix} 0 & 0 & i \\ 0 & 0 & 0 \\ -i & 0 & 0 \end{pmatrix}$,

$L_3 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \Rightarrow e^{-i\vec{\alpha} \cdot \vec{L}}$ is a rotation by angle $|\vec{\alpha}|$ around $\vec{\alpha}$-direction.

$\Rightarrow H$ is invariant under $\vec{M} \to \vec{M}' = e^{-i\vec{\alpha} \cdot \vec{L}} \vec{M}$.

$\Rightarrow$ ground state is not rotationally symmetric:

$R|0\rangle \neq |0\rangle \Rightarrow$ if $R = e^{i\vec{\alpha} \cdot \vec{Q}}$, $\vec{Q}$ ~ conserved charges of symmetry $\Rightarrow Q^i |0\rangle \neq 0$ $\left(\begin{matrix}\text{equivalently} \\ \langle 0|\vec{M}|0\rangle \neq 0\end{matrix}\right)$

General Discussion

Imagine a system with Hamiltonian H and conserved symmetry charges Q^i : $[H, Q^i] = 0$.

Act on vacuum: $H|0\rangle = 0$ (can choose vacuum to be 0-energy state)

$$H Q^i|0\rangle = \underbrace{[H, Q^i]}_{=0}|0\rangle + Q^i \underbrace{H|0\rangle}_{=0} = 0$$

$\Rightarrow H Q^i|0\rangle = 0 \Rightarrow$ either

(i) $Q^i|0\rangle = 0$ ~ no broken symmetries, vacuum is invariant under Q^i: $e^{i\vec{\varepsilon}\cdot\vec{Q}}|0\rangle = |0\rangle$.

(ii) $Q^i|0\rangle \neq 0 \Rightarrow$ vacuum is degenerate, more than one state such that $H|\psi_0\rangle = 0$.

(e.g. rotating ground state in L-G model would give other possible ground states)

$\Rightarrow$ if the system spontaneously chooses one of these $|\psi_0\rangle$ states for its ground state $\Rightarrow$ spontaneous symmetry breaking.

The Nambu - Goldstone Theorem.

Theorem Spontaneous breakdown of a continuous symmetry implies existence of massless spinless particles. (Nambu - Goldstone bosons)

(Nambu '60, Goldstone '61)

Proof j_μ is a conserved current $\partial_\mu j^\mu = 0$, due to some symmetry

$Q(t) = \int d^3x \; j_0(\vec{x}, t)$ is the conserved charge.

For generic field $\varphi(x)$:

$$\varphi(x) \to \varphi'(x) = e^{i\alpha Q} \varphi(x) e^{-i\alpha Q} = \varphi(x) +$$
$$+ i\alpha [Q, \varphi] + \dots$$

$$\Rightarrow 0 = \int d^3x \left[\partial_\mu j^\mu(\vec{x}, t), \varphi(0) \right] = \partial_0 \int d^3x \left[j^0(\vec{x}, t), \varphi(0) \right]$$

+ spatial surface term

$$\Rightarrow \frac{d}{dt} \left[Q(t), \varphi(0) \right] = 0 \Rightarrow \langle 0 | \left[Q(t), \varphi(0) \right] | 0 \rangle = v \neq 0$$

with v time-independent (constant) quantity

$$v = \langle 0 | \left[Q(t), \varphi(0) \right] | 0 \rangle = \langle 0 | Q(t) \varphi(0) | 0 \rangle -$$

$$- \langle 0 | \varphi(0) Q(t) | 0 \rangle = \int d^3x \left[\langle 0 | j_0(t, \vec{x}) \varphi(0) | 0 \rangle \right.$$

$$- \langle 0 | \varphi(0)\, j_0(t,\vec{x}) | 0 \rangle \Big]$$

Insert a complete set of intermediate states

$$\mathbb{1} = \sum_n |n\rangle\langle n| \quad \Rightarrow \text{get}$$

$$v = \sum_n \int d^3x \Big[\langle 0 | j_0(t,\vec{x}) | n\rangle \langle n | \varphi(0) | 0\rangle -$$

$$- \langle 0 | \varphi(0) | n\rangle \langle n | j_0(t,\vec{x}) | 0\rangle \Big]$$

Now, in Heisenberg picture one can write

$$j_0(t,\vec{x}) = j_0(x) = e^{i\hat{p}\cdot x}\, j_0(0)\, e^{-i\hat{p}\cdot x}$$

where $\hat{p}^\mu \overset{=(H,\vec{P})}{\text{is}}$ the 4-momentum operator, $\hat{p}^\mu|0\rangle = 0$

$$\Rightarrow v = \int d^3x \sum_n \Big[\langle 0 | e^{i\hat{p}\cdot x}\, j_0(0)\, e^{-i\hat{p}\cdot x} | n\rangle \langle n | \varphi(0) | 0\rangle$$

$$- \langle 0 | \varphi(0) | n\rangle \langle n | e^{i\hat{p}\cdot x}\, j_0(0)\, e^{-i\hat{p}\cdot x} | 0\rangle \Big]$$

Take $\langle 0 | e^{i\hat{p}\cdot x}\, j_0(0)\, e^{-i\hat{p}\cdot x} | n\rangle = \langle 0 | j_0(0) | n\rangle \cdot e^{-ip_n\cdot x}$

$$\Rightarrow v = (2\pi)^3 \sum_n \delta^3(\vec{p}_n) \Big[e^{-iE_n t} \langle 0 | j_0(0) | n\rangle \cdot$$

$$\cdot \langle n | \varphi(0) | 0\rangle - \langle 0 | \varphi(0) | n\rangle \langle n | j_0(0) | 0\rangle\, e^{iE_n t} \Big].$$

LHS $= v =$ time-independent (constant), $v \neq 0$ as SSB makes it $\neq 0$

integrate over time: $\lim_{T\to\infty} \int_{-T}^{T} dt$ $\Rightarrow$

$$\Rightarrow \lim_{T\to\infty} (2T)\cdot v = (2\pi)^4 \sum_n \delta^3(\vec{p}_n)\, \delta(E_n) \Big[\langle 0|j_0(0)|n\rangle \cdot$$

$$\cdot \langle n|\varphi(0)|0\rangle - \langle 0|\varphi(0)|n\rangle \langle n|j_0(0)|0\rangle \Big]$$

as $\lim_{T\to\infty} \int_{-T}^{T} dt = 2\pi\, \delta(0)$

$$\Rightarrow 2\pi\, \delta(0)\, v = (2\pi)^4 \sum_n \delta^3(\vec{p}_n)\, \delta(E_n) \Big[\langle 0|j_0(0)|n\rangle \cdot$$

$$\cdot \langle n|\varphi(0)|0\rangle - \langle 0|\varphi(0)|n\rangle \langle n|j_0(0)|0\rangle \Big]$$

$\Rightarrow$ at $\vec{p}_n = 0$ have a spectrum of states E_n

$\Rightarrow$ for equation to hold need to have at least one state with $E_n = 0$ $\Rightarrow$ get $\delta(0)$ too. (all other E_n's give zero contributions)

$\Rightarrow$ there <u>must</u> be a state with $E_n = 0$, $\vec{p}_n = 0$

$\Rightarrow$ <u>a massless particle</u> (Goldstone boson)

(or Nambu - Goldstone boson)

$\Rightarrow \langle n|\varphi(0)|0\rangle \neq 0$, $\langle 0|j_0(0)|n\rangle \neq 0$ for state $|n\rangle$

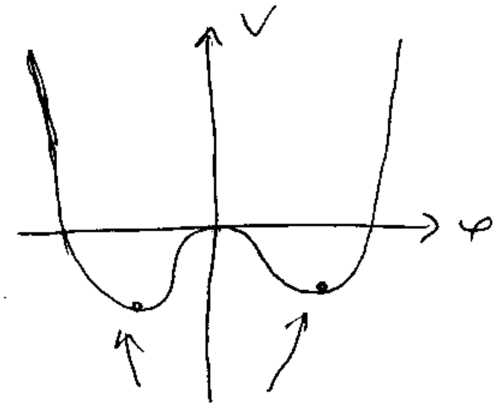
$\Rightarrow \varphi \sim$ scalar field $\Rightarrow$ boson.

Example 1: $\varphi \sim$ real scalar field

$$\mathcal{L} = \frac{1}{2}\partial_\mu \varphi \partial^\mu \varphi + \underbrace{\frac{\mu^2}{2}\varphi^2 - \frac{\lambda}{4}\varphi^4}_{-V(\varphi)} \Rightarrow \varphi \to -\varphi \text{ symmetric}$$

$\mu^2 > 0 \Rightarrow$ symmetry is broken:

vacuum: $\mu^2 \cdot \varphi - \lambda \varphi^3 = 0$



two vacua

$$\Rightarrow \varphi = \pm v = \pm \mu \cdot \sqrt{\frac{1}{\lambda}}$$

$\Rightarrow$ once the system picks $+v$ or $-v$

the $\varphi \to -\varphi$ is spontaneously broken.

$\Rightarrow$ Near the vacuum at v write $\varphi = v + \varphi'$

$$\Rightarrow \mathcal{L} = \frac{1}{2}\partial_\mu \varphi' \partial^\mu \varphi' + \frac{\mu^2}{2}(v+\varphi')^2 - \frac{\lambda}{4}(v+\varphi')^4 =$$

$$= (\text{drop constants}) = \frac{1}{2}\partial_\mu \varphi' \partial^\mu \varphi' + \underbrace{\cancel{(\mu^2 - \lambda v^2)}}_{0}\varphi' v$$

$$- \underbrace{\frac{\lambda}{4} 6 v^2}_{\mu^2 \frac{3}{2}} \varphi'^2 - \frac{\lambda}{4}\cdot 4 v \varphi'^3 + \frac{\mu^2}{2}\varphi'^2 - \frac{\lambda}{4}\varphi'^4 =$$

$$= \frac{1}{2}\partial_\mu \varphi' \partial^\mu \varphi' - \mu^2 \varphi'^2 - \lambda v \varphi'^3 - \frac{\lambda}{4}\varphi'^4$$

$\Rightarrow \varphi'$ has mass $= \mu \cdot \sqrt{2}$ not massless!

$\Rightarrow$ Is Goldstone theorem wrong? No, it's just that $\varphi \to -\varphi$ symmetry is discrete!

(G. th'm is about continuous symmetries.)

Example 2: Abelian σ-Model:

$$\mathcal{L} = \frac{1}{2}\,\partial_\mu \sigma\,\partial^\mu \sigma + \frac{1}{2}\,\partial_\mu \pi\,\partial^\mu \pi + \underbrace{\frac{\mu^2}{2}(\sigma^2+\pi^2) - \frac{\lambda}{4}(\sigma^2+\pi^2)^2}_{-V}$$

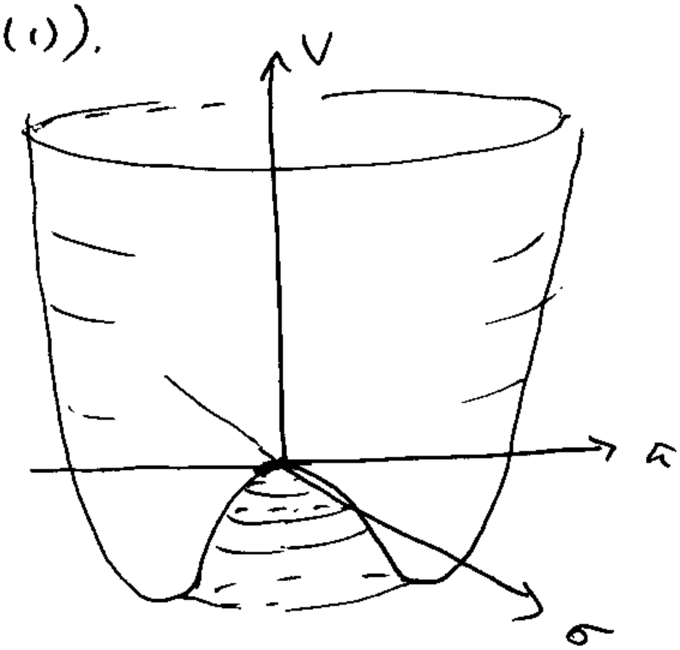
with $\mu^2 > 0$, $\lambda > 0$ (Constants).

σ, π ~ real fields

$\mathcal{L}$ is invariant under rotations:

$$\begin{pmatrix} \sigma \\ \pi \end{pmatrix} \to \begin{pmatrix} \sigma' \\ \pi' \end{pmatrix} = \begin{pmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{pmatrix}\begin{pmatrix} \sigma \\ \pi \end{pmatrix}, \quad \alpha \sim \text{real \#}$$

$\Rightarrow$ $O(2)$ symmetry $(= U(1))$.



$\Rightarrow$ get "Mexican hat" potential again.

$\Rightarrow$ the minimum is at

$\sigma^2 + \pi^2 = v^2$

$$\Rightarrow \left(\frac{\mu^2}{2}\cdot v^2 - \frac{\lambda}{4}\,v^4\right)'_v = 0$$

$\mu^2 \cdot v - \lambda v^3 = 0$

$$\Rightarrow v = \mu\sqrt{\frac{1}{\lambda}}$$

$\Rightarrow$ Direction in (σ, π) space is random $\Rightarrow$

$\Rightarrow$ pick the vacuum to be at $\langle 0|\sigma|0\rangle = v$, $\langle 0|\pi|0\rangle = 0$.

Expand $\mathcal{L}$ near the vacuum: $\sigma = v + \sigma'$

$$\Rightarrow \mathcal{L} = \frac{1}{2}\partial_\mu \sigma' \partial^\mu \sigma' + \frac{1}{2}\partial_\mu \vec{\pi}\, \partial^\mu \vec{\pi} + \frac{\mu^2}{2}\left[(v+\sigma')^2 + \vec{\pi}^2\right]$$

$$- \frac{\lambda}{4}\left[(v+\sigma')^2 + \vec{\pi}^2\right]^2 = \frac{1}{2}\partial_\mu \sigma' \partial_\mu \sigma' + \frac{1}{2}\partial_\mu \vec{\pi}\, \partial^\mu \vec{\pi} + \underbrace{\text{const}}_{\text{drop}}$$

$$+ \sigma' \cancel{\left[\mu^2 v - \frac{\lambda}{4}\cdot 4 v^3\right]}^{\to 0} + \sigma'^2\left[\cancel{\frac{\mu^2}{2}} - \frac{\lambda}{4}\cdot(\cancel{2v^2} + 4v^2)\right] +$$

$$+ \vec{\pi}^2 \cancel{\left[\frac{\mu^2}{2} - \frac{\lambda}{4}\cdot 2v^2\right]}^{\to 0} - \frac{\lambda}{4}\left[4\sigma' v(\sigma'^2 + \vec{\pi}^2) + (\sigma'^2 + \vec{\pi}^2)^2\right]$$

$$= \frac{1}{2}\partial_\mu \sigma' \partial_\mu \sigma' + \frac{1}{2}\partial_\mu \vec{\pi}\, \partial^\mu \vec{\pi} - \mu^2 \sigma'^2 - \lambda v\, \sigma'(\sigma'^2 + \vec{\pi}^2)$$

$$- \frac{\lambda}{4}(\sigma'^2 + \vec{\pi}^2)^2 .$$

$\Rightarrow$ now $\vec{\pi}$'s have no $\vec{\pi}^2$ term $\Rightarrow$ $\vec{\pi}$ field is <u>massless</u> in agreement with Goldsone th'm!

<u>Non-Abelian σ-Model</u>

Let's illustrate how the chiral $SU(3)_L \otimes SU(3)_R$ symmetry is broken in QCD. As an example consider breaking of $SU(2)_L \otimes SU(2)_R$ symmetry. Start with the Lagrangian:

$$\mathcal{L} = \frac{1}{2}\left(\partial_\mu \sigma \partial^\mu \sigma + \partial_\mu \vec{\pi}\, \partial^\mu \vec{\pi}\right) + \frac{\mu^2}{2}\left(\sigma^2 + \vec{\pi}^2\right) - \frac{\lambda}{4}\left[\sigma^2 + \vec{\pi}^2\right]^2$$

$\mu^2, \lambda > 0$, $\overset{\text{isoscalar}}{\sigma}$, $\vec{\pi} = \underbrace{(\pi_1, \pi_2, \pi_3)}_{\text{iso triplet, pions}}$ ~ real fields

Define a 2×2 matrix field $\Sigma = \sigma \mathbb{1} + i\,\vec{\tau}\cdot\vec{\pi}$

$\tau^1, \tau^2, \tau^3 \sim$ Pauli matrices (we use τ to not confuse them with σ)

$$\Rightarrow \operatorname{tr}\left[\Sigma \quad \Sigma^+\right] = \operatorname{tr}\left[\sigma^2 \mathbb{1} + i\,\vec{\tau}\cdot\vec{\pi}\,(-i)\,\vec{\tau}\cdot\vec{\pi}\right]$$

$$= 2\sigma^2 + 2\vec{\pi}^2 \qquad \text{as } \operatorname{tr} \tau^i\tau^j = 2\delta^{ij}$$

$$\Rightarrow \operatorname{tr}\left[\partial_\mu \Sigma \;\; \partial^\mu \Sigma^+\right] = 2\left[\partial_\mu\sigma\,\partial^\mu\sigma + \partial_\mu\vec{\pi}\;\;\partial^\mu\vec{\pi}\right]$$

$$\Rightarrow \mathcal{L}_\Sigma = \frac{1}{4}\left[\operatorname{tr}\,\partial_\mu\Sigma\;\;\partial^\mu\Sigma^+\right] + \frac{\mu^2}{4}\operatorname{tr}\left[\Sigma\Sigma^+\right] - \frac{\lambda}{16}\left(\operatorname{tr}\left[\Sigma\Sigma^+\right]\right)^2$$

Now add "quarks": (originaly they were protons and neutrons): $q = \begin{pmatrix} u \\ d \end{pmatrix}$ or $\begin{pmatrix} p \\ n \end{pmatrix} = q^N$

$$\mathcal{L} = \bar{q}^N i\gamma\cdot\partial q^N - g\,\bar{q}^N\left[\sigma\mathbb{1} + i\,\vec{\tau}\cdot\vec{\pi}\,\gamma_5\right]q^N + \mathcal{L}_\Sigma$$

Such that

$$\mathcal{L} = \bar{q}^N i\gamma\cdot\partial q^N - g\,\bar{q}^N\left[\sigma\mathbb{1} + i\,\vec{\tau}\cdot\vec{\pi}\,\gamma_5\right]q^N +$$
$$+ \frac{1}{2}\left(\partial_\mu\sigma\,\partial^\mu\sigma + \partial_\mu\vec{\pi}\;\;\partial^\mu\vec{\pi}\right) + \frac{\mu}{2}\left(\sigma^2 + \vec{\pi}^2\right) - \frac{\lambda}{4}\left(\sigma^2 + \vec{\pi}^2\right)^2$$

full Lagrangian for $SU(2)_L \otimes SU(2)_R$ σ-model.

(Gell-Mann & Levi, 1960)

As usual write $q^N = q_L^N + q_R^N \Rightarrow$

$$\bar{q}^N i\gamma\cdot\partial q^N = \bar{q}_L^N i\gamma\cdot\partial q_L^N + \bar{q}_R^N i\gamma\cdot\partial q_R^N$$