Last time: talked about SSB: $Q^i|0\rangle \neq 0$ ~ degenerate vacua

Nambu-Goldstone th'm

~~continuous symmetry~~ SSB $\Rightarrow$ massless spinless particles (Nambu - Goldstone bosons)

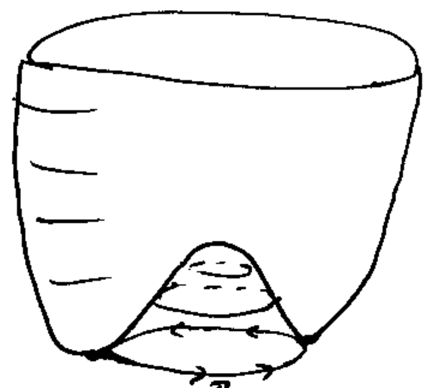
Simple interpretation: ~ suppose minimum of V is degenerate, has flat directions:

$\Rightarrow$ expand $\mathcal{L}$ near that min $\Rightarrow$

$\Rightarrow$ flat directions would give

Nambu - Goldstone massless modes

flat directions (# symm. broken) = # N.-G. modes



(e.g. bent stick (~SSB, but

rotational modes of the string are "flat" ~ Goldstone modes)

$\Rightarrow$ worked out examples: discrete symmetry ~ no G. modes

Non-Abelian σ-Model (cont'd)

$$\boxed{\mathcal{L} = \bar{q}^N i\gamma\cdot\partial q^N + \frac{1}{2}\left(\partial_\mu\sigma\,\partial^\mu\sigma + \partial_\mu\vec{\pi}\cdot\partial^\mu\vec{\pi}\right) + \frac{\mu}{2}\left(\sigma^2+\vec{\pi}^2\right) - \frac{\lambda}{4}\left(\sigma^2+\vec{\pi}^2\right)^2 - g\,\bar{q}^N\left[\mathbb{1}\sigma + i\,\vec{\tau}\cdot\vec{\pi}\,\gamma_5\right]q^N}$$

Gell-Mann, Levi '60

$q^N = \begin{pmatrix} p \\ n \end{pmatrix}$ ~ proton / neutron, $\vec{\pi} = (\pi^1, \pi^2, \pi^3)$ ~ pions, σ ~ auxiliary scalar field.

defined $\Sigma = \mathbb{1}\sigma + i\,\vec{\tau}\cdot\vec{\pi} \Rightarrow tr[\Sigma\Sigma^+] = 2(\sigma^2 + \vec{\pi}^2)$

$$\Rightarrow \mathcal{L}_{pions} = \frac{1}{4}\, tr[\partial_\mu\Sigma\, \partial^\mu\Sigma^+] + \frac{\mu^2}{4}\, tr[\Sigma\Sigma^+] - \frac{\lambda}{16}\left(tr[\Sigma\Sigma^+]\right)^2.$$

what about nucleons? interaction term?

σ-Model is an example of effective lagrangian, it is a "reduction" of the full QCD lagrangian to the low-energy regime with interactions carried by pions and the fermions being protons & neutrons.

Define a 2×2 matrix field $\Sigma = \sigma \mathbb{1} + i\,\vec{\tau}\cdot\vec{\pi}$

τ^1, τ^2, τ^3 ~ Pauli matrices (we use τ to not confuse them with σ)

$$\Rightarrow \operatorname{tr}\left[\Sigma \quad \Sigma^+\right] = \operatorname{tr}\left[\sigma^2 \mathbb{1} + i\,\vec{\tau}\cdot\vec{\pi}(-i)\,\vec{\tau}\cdot\vec{\pi}\right]$$

$$= 2\sigma^2 + 2\vec{\pi}^2 \quad \text{as } \operatorname{tr}\tau^i\tau^j = 2\delta^{ij}$$

$$\Rightarrow \operatorname{tr}\left[\partial_\mu \Sigma\ \partial^\mu \Sigma^+\right] = 2\left[\partial_\mu\sigma\,\partial^\mu\sigma + \partial_\mu\vec{\pi}\ \partial^\mu\vec{\pi}\right]$$

$$\Rightarrow \mathcal{L}_\Sigma = \frac{1}{4}\left[\operatorname{tr}\ \partial_\mu\Sigma\ \partial^\mu\Sigma^+\right] + \frac{\mu^2}{4}\operatorname{tr}\left[\Sigma\Sigma^+\right] - \frac{\lambda}{16}\left(\operatorname{tr}\left[\Sigma\Sigma^+\right]\right)^2$$

Now add "quarks": (originaly they were protons and neutrons): $q = \begin{pmatrix} u \\ d \end{pmatrix}$ or $\begin{pmatrix} p \\ n \end{pmatrix} = q^N$

$$\mathcal{L} = \bar{q}^N i\gamma\cdot\partial q^N - g\,\bar{q}^N\left[\sigma\mathbb{1} + i\,\vec{\tau}\cdot\vec{\pi}\,\gamma_5\right]q^N + \mathcal{L}_\Sigma$$

Such that

$$\mathcal{L} = \bar{q}^N i\gamma\cdot\partial q^N - g\,\bar{q}^N\left[\sigma\mathbb{1} + i\,\vec{\tau}\cdot\vec{\pi}\,\gamma_5\right]q^N + \frac{1}{2}\left(\partial_\mu\sigma\,\partial^\mu\sigma + \partial_\mu\vec{\pi}\ \partial^\mu\vec{\pi}\right) + \frac{\mu}{2}\left(\sigma^2 + \vec{\pi}^2\right) - \frac{\lambda}{4}\left(\sigma^2 + \vec{\pi}^2\right)^2$$

full Lagrangian for $SU(2)_L \otimes SU(2)_R$ σ-model.

(Gell-Mann & Levi, 1960)

As usual write $q^N = q^N_L + q^N_R \Rightarrow$

$$\bar{q}^N i\gamma\cdot\partial q^N = \bar{q}^N_L i\gamma\cdot\partial q^N_L + \bar{q}^N_R i\gamma\cdot\partial q^N_R$$

$$\bar{q}^N\left[\sigma \mathbb{1} + i\vec{\tau}\cdot\vec{\pi}\,\gamma_5\right] q^N = \Big(\overbrace{\bar{q}^N \frac{\mathbb{1}+\gamma_5}{2}}^{\bar{q}_L} + \overbrace{\bar{q}^N\frac{\mathbb{1}-\gamma_5}{2}}^{\bar{q}_R}\Big)\cdot$$

$$\cdot\left[\sigma \mathbb{1} + i\,\vec{\tau}\cdot\vec{\pi}\,\gamma_5\right]\Big(\underbrace{\frac{\mathbb{1}-\gamma_5}{2} q^N}_{q_L} + \underbrace{\frac{\mathbb{1}+\gamma_5}{2} q^N}_{q_R}\Big) \quad = \Big|^{\text{as } (\gamma_5)^2=1}$$

$$= \sigma\left[\bar{q}_L^N q_R^N + \bar{q}_R^N q_L^N\right] + i \left[-\bar{q}_R^N\,\vec{\tau}\cdot\vec{\pi}\, q_L^N + \bar{q}_L^N\,\vec{\tau}\cdot\vec{\pi}\, q_R^N\right]$$

$$= \bar{q}_L^N \Sigma\, q_R^N + \bar{q}_R^N \Sigma^+ q_L^N$$

$$\Rightarrow \boxed{\begin{aligned}\mathcal{L} &= \bar{q}_L^N\, i\gamma\cdot\partial\, q_L^N + \bar{q}_R^N\, i\gamma\cdot\partial\, q_R^N + \frac{1}{4}\operatorname{tr}\left[\partial_\mu\Sigma\,\partial^\mu\Sigma^+\right] \\ &+ \frac{\mu^2}{4}\operatorname{tr}\left[\Sigma\Sigma^+\right] - \frac{\lambda}{16}\left(\operatorname{tr}\left[\Sigma\Sigma^+\right]\right)^2 - \\ &- g\left[\bar{q}_L^N\Sigma\, q_R^N + \bar{q}_R^N\Sigma^+ q_L^N\right]\end{aligned}}$$

(effective low-energy lagrangian not QCD, but has the right symmetries)

$\Rightarrow$ this Lagrangian is symmetric under

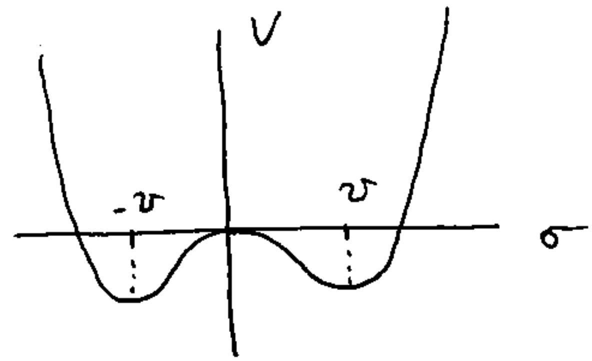
$$\psi_L \to \psi_L' = e^{i\vec{\alpha}_L\cdot\frac{\vec{\tau}}{2}}\psi_L \equiv U_L\psi_L$$

$$\psi_R \to \psi_R' = e^{i\vec{\alpha}_R\cdot\frac{\vec{\tau}}{2}}\psi_R \equiv U_R\psi_R$$

$$\Sigma \to \Sigma' = U_L\,\Sigma\, U_R^+$$

$\Rightarrow$ it has $SU(2)_L \otimes SU(2)_R$ symmetry!

For $m^2 > 0$ the $SU(2)_L \otimes SU(2)_R$ symmetry is spontaneously broken:



$$\left(\frac{\mu^2}{2}\sigma^2 - \frac{\lambda}{4}\sigma^4\right)' = 0$$

$$\Rightarrow \quad v = \frac{\mu}{\sqrt{\lambda}}$$

$\Rightarrow$ pick $\langle 0|\sigma|0\rangle = v$, $\langle 0|\vec{\pi}|0\rangle = 0$ as the vacuum.

Write $\sigma = v + \sigma'$ $\Rightarrow$ $\mathcal{L} = \bar{q}^N i\gamma\cdot\partial q^N - g\,\bar{q}^N[v + \sigma' + i\vec{\tau}\cdot\vec{\pi}\gamma_5]q^N + \frac{1}{2}\left[\partial_\mu\sigma'\partial^\mu\sigma' + \partial_\mu\vec{\pi}\partial^\mu\vec{\pi}\right] - \mu^2\sigma'^2 - \lambda v\sigma'(\sigma'^2 + \vec{\pi}^2) - \frac{\lambda}{4}\left(\sigma'^2 + \vec{\pi}^2\right)^2.$

$\Rightarrow$ σ' has mass $\sqrt{2}\,\mu$.

$\vec{\pi}$ have mass 0. ~ Goldstone bosons (pions)

q^N (proton, neutron) have mass gv. ~ can be large!

Identify $\vec{\pi} \Leftrightarrow \bar{q}\gamma_5\vec{\tau}q$ (q now are real quarks)

$\sigma \Leftrightarrow \bar{q}q$

q^N ~ proton, neutron ~ nucleons

$\Rightarrow$ $SU(2)_L \otimes SU(2)_R$ is spontaneously broken down to $SU(2)$

$\Rightarrow$ pions (π^+, π^0, π^-) are Goldstone bosons of chiral SSB, $m_\pi = 0$ ($SU(2)$ has 3 generators $\Rightarrow$ 3 pions!)

$\Rightarrow$ protons, neutrons get a mass $m_N = g v$ which is large.

$\Rightarrow$ if $SU(2)_L \otimes SU(2)_R$ was exact would have $m_\pi = 0$ but as $m_u \neq m_d \neq 0$ $SU(2)_L \otimes SU(2)_R$ is explicitly broken too $\Rightarrow$ get massive pions!

$\Rightarrow$ for $N_f = 3$ have $SU(3)_L \otimes SU(3)_R$ broken down spontaneously to $SU(3)$ flavor.

$\Rightarrow$ $SU(3)$ has 8 symmetry charges

$Q^a, \quad a = 1, \dots, 8$

$\Rightarrow$ have 8 Goldstone bosons:

$\pi^+, \pi^-, \pi^0, K^+, K^0, \bar{K}^0, K^-, \eta^0.$

$\Rightarrow$ $SU(3)_L \otimes SU(3)_R$ is also badly broken explicitly as $m_s \neq m_u \neq m_d \neq 0$ $\Rightarrow$ K's & η are also massive!

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what is v (VEV) in QCD? Remember $\sigma = \bar{q} q$ $\Rightarrow$

$v = \langle 0 | \bar{q} q | 0 \rangle \simeq -(230\ \text{MeV})^3$ quark condensate or chiral condensate.

$m_\pi^2 \sim (m_u + m_d) \langle 0 | \bar{q} q | 0 \rangle.$