Last time: we reviewed 4-vectors:

contravariant:

$$A'^{\mu} = \frac{\partial x'^{\mu}}{\partial x^{\nu}} A^{\nu}$$

covariant:

$$B'_{\mu} = \frac{\partial x^{\nu}}{\partial x'^{\mu}} B_{\nu}$$

defined scalar product $A_{\mu} B^{\mu}$, metric $g_{\mu\nu} = \begin{pmatrix} 1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix}$

$$A^{\mu} = g^{\mu\nu} A_{\nu}, \quad A_{\mu} = g_{\mu\nu} A^{\nu}, \quad g_{\mu\alpha} g^{\alpha\nu} = \delta_{\mu}^{\nu}.$$

defined derivatives $\partial_{\mu} = \frac{\partial}{\partial x^{\mu}}$, $\partial^{\mu} = \frac{\partial}{\partial x_{\mu}}$

started talking about classical field theory

for Free Scalar Field: φ ~ scalar field

the action $\boxed{S = \int d^4x \, \mathcal{L}(\varphi, \partial_{\mu}\varphi)}$

lagrangian density

least action principle $\delta S = 0 \Rightarrow$

$$\boxed{\frac{\delta \mathcal{L}}{\delta \varphi} - \partial_{\mu} \frac{\delta \mathcal{L}}{\delta(\partial_{\mu}\varphi)} = 0}$$

Euler-Lagrange equations

for massive scalar field:

$$\boxed{\mathcal{L} = \frac{1}{2} \partial_{\mu}\varphi \, \partial^{\mu}\varphi - \frac{m^2}{2}\varphi^2}$$

$\Rightarrow$ EOM is $\boxed{\left[\partial_{\mu}\partial^{\mu} + m^2\right]\varphi = 0}$ Klein-Gordon equation

or $\left[\Box + m^2\right]\varphi = 0$

$$\Rightarrow \boxed{\left[\partial_\mu \partial^\mu + m^2\right]\varphi = 0}$$ Klein-Gordon equation

or $$\left[\Box + m^2\right]\varphi = 0$$

To solve K-G equation write $\varphi(x) = \int d^4k \, e^{-ik\cdot x}\tilde{\varphi}(k)$

with $k \cdot x = k_\mu x^\mu = k^0 x^0 - \vec{k}\cdot\vec{x}$.

$$\left[\Box + m^2\right]\varphi = \int d^4k \, \tilde{\varphi}(k)\left(\Box + m^2\right) e^{-ik\cdot x} = \int d^4k \, \tilde{\varphi}(k) \cdot$$

$$\cdot \left[-k^2 + m^2\right] = 0 \quad \text{with } k^2 = k_\mu k^\mu = (k^0)^2 - (\vec{k})^2.$$

$\Rightarrow \left[k^2 - m^2\right]\tilde{\varphi} = 0 \Rightarrow$ as $\tilde{\varphi} \neq 0 \Rightarrow k^2 = m^2$ or

$\mathcal{E}_k^2 - \vec{k}^2 = m^2 \Rightarrow \mathcal{E}_k = \pm\sqrt{\vec{k}^2 + m^2} \Rightarrow$ define $\mathcal{E}_k = \sqrt{\vec{k}^2 + m^2}$

$$\Rightarrow \varphi(x) = \int \frac{d^3k}{(2\pi)^3\, 2\mathcal{E}_k}\left[a_{\vec{k}}\, e^{-i\mathcal{E}_k t + i\vec{k}\cdot\vec{x}} + a^*_{\vec{k}}\, e^{i\mathcal{E}_k t - i\vec{k}\cdot\vec{x}}\right]$$

most general solution.

Canonical Quantization

In your QM class you must have seen that if we treat K-G equation as the equation for a single-particle wave function $\varphi(x)$ (just like

Schroedinger eqn, except now relativistic),

then we have a hoard of problems:

(i) as the energy $= \pm\sqrt{\vec{k}^2+m^2}$ => can have free particles with negative energy!?

(ii) particle propagation $\langle\vec{x}|e^{-it\hat{H}}|\vec{y}\rangle$ from point $\vec{y}$ to pt. $\vec{x}$ is <u>acausal</u> ~ can find the particle outside the light-cone => it would propagate "faster" than light...

~ in general we know that relativistic kinematics allows for a particle to decay into several particles => particle # is not conserved => should not have a single-particle wave function interpretation.

=> we quantize the system treating $\varphi(x)$ as a <u>field</u>!

Again, let us draw an analogy with Quantum Mechanics. Start with a system with degrees of freedom q_i described by Lagrangian $L(q_i, \dot{q}_i)$ $i=1,\ldots,N$. Define <u>canonical momenta</u> $p_i = \frac{\partial L}{\partial \dot{q}_i}$

$\Rightarrow$ get Hamiltonian $H = \sum_i \dot{q}_i p_i - L \Rightarrow H(q_i, p_i)$

$\Rightarrow$ quantize by "promoting" q_i, p_i to operators

$\hat{q}_i, \hat{p}_i$ such that ($[\hat{A},\hat{B}] \equiv \hat{A}\hat{B} - \hat{B}\hat{A}$ ~ commutator)

$$\boxed{[\hat{q}_i, \hat{p}_j] = i\delta_{ij}, \quad [\hat{q}_i, \hat{q}_j] = [\hat{p}_i, \hat{p}_j] = 0}$$

Mechanics		Field Theory
q_i	$\rightarrow$	$\varphi(x)$
i	$\rightarrow$	x^μ
$p_i = \dfrac{\partial L}{\partial \dot{q}_i}$	$\rightarrow$	$\pi(x) = \dfrac{\delta \mathcal{L}}{\delta \dot{\varphi}(x)}$
$[\hat{q}_i, \hat{p}_j] = i\delta_{ij}$	$\rightarrow$	$[\varphi(\vec{x},t), \pi(\vec{x}',t)] = i\delta(\vec{x}-\vec{x}')$
$[\hat{q}_i, \hat{q}_j] = 0$	$\rightarrow$	$[\varphi(\vec{x},t), \varphi(\vec{x}',t)] = 0$
$[\hat{p}_i, \hat{p}_j] = 0$	$\rightarrow$	$[\pi(\vec{x},t), \pi(\vec{x}',t)] = 0$.

($\Rightarrow$ Note that canonical quantization favors time direction and is therefore not relat. inv. (physics is indeed invariant).)

$\hat{H}(\hat{q}_i, \hat{p}_i) \quad \rightarrow \quad \hat{H} = \int d^3x \cdot \mathcal{H}(\varphi, \pi)$

Hamiltonian gives time-evolution of the system

We have:

$$\boxed{\begin{aligned}&[\varphi(\vec{x},t),\ \pi(\vec{x}',t)] = i\,\delta(\vec{x}-\vec{x}')\\ &[\varphi(\vec{x},t),\ \varphi(\vec{x}',t)] = [\pi(\vec{x},t),\ \pi(\vec{x}',t)] = 0\end{aligned}}$$

where φ, π are operators, $\boxed{\pi(x) \equiv \dfrac{\delta \mathcal{L}}{\delta \dot{\varphi}(x)}}$

Write

$$\varphi(x) = \int \frac{d^3k}{(2\pi)^3 2\varepsilon_k}\left[\hat{a}_{\vec{k}}\, e^{-ik\cdot x} + \hat{a}^{+}_{\vec{k}}\, e^{ik\cdot x}\right] \quad (\text{now } \hat{a}, \hat{a}^{+} \text{ are } \underline{\text{operators}})$$

$$\Rightarrow \boxed{\begin{aligned}&[\hat{a}_{\vec{k}},\ \hat{a}^{+}_{\vec{k}'}] = (2\pi)^3\, 2\varepsilon_k\, \delta(\vec{k}-\vec{k}')\\ &[\hat{a}_{\vec{k}},\ \hat{a}_{\vec{k}'}] = [\hat{a}^{+}_{\vec{k}},\ \hat{a}^{+}_{\vec{k}'}] = 0\end{aligned}} \Leftarrow \begin{cases} \text{for } \mathcal{L} = \frac{1}{2}\partial_\mu\varphi\,\partial^\mu\varphi + \ldots \\ \pi = \partial_0\varphi = \dot{\varphi}\end{cases}$$

(can show)

The Hamiltonian is $\boxed{H = \int d^3x\left[\dot{\varphi}(x)\,\pi(x) - \mathcal{L}\right]}$

$$\Rightarrow \quad i\frac{\partial}{\partial t}\hat{O} = [\hat{O}, \hat{H}] \quad \text{Heisenberg picture.}$$

$$\Rightarrow \hat{O}(\vec{x},t) = e^{i\hat{H}t}\,\hat{O}(\vec{x},0)\,e^{-i\hat{H}t}$$

<u>Example</u>: $\mathcal{L} = \frac{1}{2}\partial_\mu\varphi\,\partial^\mu\varphi - \frac{m^2}{2}\varphi^2 \quad \Rightarrow$

$$\Rightarrow \pi(x) = \frac{\delta\mathcal{L}}{\delta\dot{\varphi}} = \frac{\delta\mathcal{L}}{\delta(\partial_0\varphi)} = \partial_0\varphi \quad \Rightarrow$$

$$H = \int d^3x\left[\dot{\varphi}^2 - \frac{1}{2}\dot{\varphi}^2 + \frac{1}{2}(\vec{\nabla}\varphi)^2 + \frac{m^2}{2}\varphi^2\right]$$

define $\overleftrightarrow{\partial_0}$ by $\varphi_1 \overleftrightarrow{\partial_0} \varphi_2 = \varphi_1 \partial_0 \varphi_2 - \varphi_2 \partial_0 \varphi_1$

Note that $\int d^3x \; e^{ik\cdot x} \overleftrightarrow{\partial_0} e^{-ik'\cdot x} = \int d^3x \left[-i\mathcal{E}_{k'} - i\mathcal{E}_k\right]$

$\cdot e^{ix\cdot(k-k')} = -2i\mathcal{E}_k (2\pi)^3 \delta(\vec{k}-\vec{k}')$

$\Rightarrow$ if $\varphi(x) = \int \frac{d^3k}{(2\pi)^3 2\mathcal{E}_k} \left[\hat{a}_{\vec{k}} e^{-ik\cdot x} + \hat{a}^+_{\vec{k}} e^{ik\cdot x}\right]$

$$\Rightarrow \int d^3x \; e^{ik'\cdot x} \overleftrightarrow{\partial_0} \varphi(x) = \int \frac{d^3k}{(2\pi)^3 2\mathcal{E}_k} \hat{a}_{\vec{k}} (-2i\mathcal{E}_k)(2\pi)^3 \delta(\vec{k}-\vec{k}')$$

$$= -i\hat{a}_{\vec{k}'}$$

as $\int d^3x \; e^{ik\cdot x} \overleftrightarrow{\partial_0} e^{+ik'\cdot x} = 0$. (why?)

$$\Rightarrow \boxed{\hat{a}_{\vec{k}} = \int d^3x \; e^{ik\cdot x} \, i\overleftrightarrow{\partial_0} \varphi(x)}$$

Similarly $\boxed{\hat{a}^+_{\vec{k}} = \int d^3x \; \varphi(x) \, i\overleftrightarrow{\partial_0} e^{-ik\cdot x}}$

$$\Rightarrow \left[\hat{a}_{\vec{k}}, \hat{a}^+_{\vec{k}'}\right] = \int d^3x \, d^3y \left[e^{ik\cdot x} i\overleftrightarrow{\partial_0}\varphi(x), \varphi(y) i\overleftrightarrow{\partial_0} e^{-ik'\cdot y}\right]$$

$$= \int d^3x \, d^3y \; e^{ik\cdot x - ik'\cdot y} (-) \left[\pi(x) - i\mathcal{E}_k \varphi(x), -i\mathcal{E}_{k'}\varphi(y) - \pi(y)\right]$$

$$= \int d^3x \, d^3y \; e^{ik\cdot x - ik'\cdot y} (-i) \left\{\left[\varphi(\overset{\vec{x},t}{x}), \pi(\overset{\vec{y},t}{y})\right]\mathcal{E}_k + \left[\varphi(y), \pi(x)\right]\mathcal{E}_{k'}\right\} =$$

$$= \int d^3x \, d^3y \; e^{ik\cdot x - ik'\cdot y} (\mathcal{E}_k + \mathcal{E}_{k'}) \delta(\vec{x}-\vec{y}) = e^{i(\mathcal{E}_k - \mathcal{E}_{k'})\cdot t} (\mathcal{E}_k + \mathcal{E}_{k'}) (2\pi)^3 \delta(\vec{k}-\vec{k}')$$

$= 2\,\varepsilon_k\,(2\pi)^3\,\delta(\vec{k}-\vec{k}')$ as advertised!

$$\Rightarrow \quad H = \int d^3x \left[\frac{1}{2}(\partial_0 \varphi)^2 + \frac{1}{2}(\vec{\nabla}\varphi)^2 + \frac{m^2}{2}\varphi^2\right].$$

plug in $\varphi = \int \frac{d^3k}{(2\pi)^3 2\varepsilon_k}\left[\hat{a}_{\vec{k}}\, e^{-ik\cdot x} + \hat{a}^+_{\vec{k}}\, e^{ik\cdot x}\right]$

to get (after some algebra)

$$H = \int \frac{d^3k}{(2\pi)^3 2\varepsilon_k}\, \frac{\varepsilon_k}{2}\left(\hat{a}^+_{\vec{k}}\,\hat{a}_{\vec{k}} + \hat{a}_{\vec{k}}\,\hat{a}^+_{\vec{k}}\right)$$

$\Rightarrow$ defining particle number operator $\hat{N}(\vec{k}) = \hat{a}^+_{\vec{k}}\,\hat{a}_{\vec{k}}$

write $\boxed{H = \int \frac{d^3k}{(2\pi)^3\, 2\varepsilon_k}\, \varepsilon_k\left[\hat{N}(\vec{k}) + \infty\right]}$

a constant, zero-point energy, not detectable experimentally

(zero point energy: consider harm. osc. in a box: $E_n = \hbar\omega(n+\frac{1}{2}) \Rightarrow E_0 = \frac{1}{2}\hbar\omega = \frac{1}{2}\hbar\frac{2\pi m}{L} \Rightarrow \sum_m E_0 = \frac{\pi}{L}\hbar\sum_{m=0}^{\infty} m = \infty$)

$$\Rightarrow \quad H \propto \varepsilon_k \cdot N(\vec{k})$$

energies of particles; # of particles

→ makes physical sense!

Free Dirac Field

Def. Dirac γ-matrices γ^μ are defined by requiring $\boxed{\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}}$.

In Weyl representation one has

$$\gamma^0 = \begin{pmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix}, \quad \gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}, \quad i = 1, 2, 3$$

where σ^i's are Pauli matrices

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Def. $\sigma_{\mu\nu} = \frac{i}{2}[\gamma_\mu, \gamma_\nu]$ One can show that these $\sigma_{\mu\nu}$ matrices are generators of Lorentz algebra.

Def. $\psi_\alpha(x)$, $\alpha = 1, 2, 3, 4$ is called a spinor if it transforms as

$$\psi(x) \to \psi'(x') = e^{-\frac{i}{4}\omega^{\mu\nu}\sigma_{\mu\nu}}\psi(x)$$

under Lorentz transformation. $\psi = \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \\ \psi_4 \end{pmatrix}$

(E.g. $\omega^{0i} = \zeta^i \sim$ boosts, $\omega^{12} = \theta^3 \sim$ rotations, etc.)

Def. $\bar{\psi}_\alpha = \psi^\dagger_\beta (\gamma_0)_{\beta\alpha}$

One can show that $\bar{\psi}(x) \to \bar{\psi}'(x') = \bar{\psi}(x)\, e^{\frac{i}{4}\omega^{\mu\nu}\sigma_{\mu\nu}}$.

Therefore $\bar{\psi}\psi \equiv \sum_{\alpha=1}^{4} \bar{\psi}_\alpha \psi_\alpha$ is a Lorentz-invariant!