Last time: worked out neutrino oscillations:

$$\boxed{P_{\nu_\mu \to \nu_e} = \sin^2(2\theta)\, \sin^2\left(\frac{1.27\, \Delta m^2\, L}{E}\right)}$$

where $\Delta m^2 = m_2^2 - m_1^2$, $L \sim$ distance traveled
$(eV)^2/c^4$ (meters)

$E \sim$ energy of neutrinos (MeV).

We also worked out how to include ν's into the Standard Model: a possible way is: define $\nu_R^e \sim$ a weak singlet

$$\Rightarrow \quad \mathcal{L}_{R.H.\nu} = -G_R \left[\bar{L}\, \tilde{\phi}\, \nu_R^e + c.c. \right] + (\mu, \tau)$$

with $L = \begin{pmatrix} \nu_e \\ e \end{pmatrix}_L$ gives neutrinos a mass

$$\boxed{m_\nu = \frac{G_R\, v}{\sqrt{2}}}$$

$\Rightarrow$ problem: to get $m_\nu \simeq 0.04\, eV$ need

$G_R \simeq 2 \times 10^{-13}$ ~ very small! is this natural?

Quantum Chromodynamics (QCD) (cont'd)

Running Coupling and Asymptotic Freedom (cont'd)

$m_f = 0$ ~ massless quarks, μ ~ cutoff in UV

$\mathcal{M}(Q^2, \mu^2, \alpha_\mu)$ ~ dimensionless observable

$\alpha_\mu = g_\mu^2/4\pi$

$\Rightarrow \mathcal{M}(Q^2/\mu^2, \alpha_\mu)$

Physics should not depend on cutoff:

$$\boxed{\mu^2 \frac{d}{d\mu^2} \mathcal{M}(Q^2/\mu^2, \alpha_\mu) = 0}$$

Def. Beta-function $\beta(\alpha_\mu) = \mu^2 \frac{d\alpha_\mu}{d\mu^2}$

$$\Rightarrow \boxed{\left[\mu^2 \frac{\partial}{\partial \mu^2} + \beta(\alpha_\mu) \frac{\partial}{\partial \alpha_\mu}\right] \mathcal{M}(Q^2/\mu^2, \alpha_\mu) = 0}$$

RG equation (Callan, Symanzik '70)

$$\rho(\alpha_\mu) = \int_{\alpha_0}^{\alpha_\mu} \frac{d\alpha'}{\beta(\alpha')} \Rightarrow \alpha(Q^2) \equiv \rho^{-1}\left(\ln \frac{Q^2}{\mu^2} + \rho(\alpha_\mu)\right)$$

solves the RG equation (running coupling)

$\alpha(\mu^2) = \alpha_\mu$: Note that $\forall$ $\mathcal{M}(\alpha(Q^2)) = \mathcal{M}(Q^2/\mu^2, \alpha_\mu)\big|_{\mu = Q}$

solves RG eqn too $\Rightarrow$ we found the solution.

automatically satisfies RG equation.

$\Rightarrow$ We have shown that running coupling $\alpha(Q^2)$ satisfies RG equation + allows any observable dependent on it to satisfy RG equation.

$\Rightarrow$ let's find $\alpha(Q^2)$: to do this need $\rho(\alpha_\mu)$.

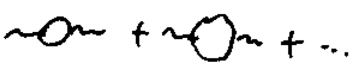
To find $\rho(\alpha_\mu)$ need $\beta(\alpha_\mu)$ ~ the beta-function. Beta-function has to be found through an explicit (hard) calculation ~ see field theory texts like Peskin.

$\Rightarrow$ in perturbation theory one usually gets:

$$\boxed{\beta(\alpha) = -\beta_2\, \alpha^2 - \beta_3\, \alpha^3 + \ldots}$$

(perturbative / small coupling α expansion)

in QCD $\quad \beta_2 = \dfrac{11\, N_c - 2\, N_f}{12\pi}$, $\quad N_c$ ~ # colors, N_f ~ # flavors



(Politzer '73, Gross & Wilczek '73)

~ was probably obtained before by 't Hooft (oral communication)

$\Rightarrow$ it is <u>very important</u> that in QCD $\beta(\alpha) < 0$ ~ beta-function is negative c.f. in QED have $\beta_2^{QED} = -\frac{1}{3\pi}$ such that

$\beta_2^{QED}(\alpha) > 0$.

$\Rightarrow$ why does this matter? Let's do the calculation at small coupling: put $\beta(\alpha) = -\beta_2 \alpha^2$

$$\Rightarrow \rho(\alpha) = \int_{\alpha_0}^{\alpha_\mu} \frac{d\alpha'}{\beta(\alpha')} = -\frac{1}{\beta_2} \int_{\alpha_0}^{\alpha_\mu} \frac{d\alpha'}{\alpha'^2} = -\frac{1}{\beta_2}\left(-\frac{1}{\alpha'}\right)\Bigg|_{\alpha_0}^{\alpha_\mu} =$$

$$= \frac{1}{\beta_2}\left(\frac{1}{\alpha_\mu} - \frac{1}{\alpha_0}\right).$$

The inverse function: $\rho(\alpha) = w \Rightarrow \alpha = \rho^{-1}(w)$

$$\Rightarrow \frac{1}{\beta_2}\left(\frac{1}{\alpha} - \frac{1}{\alpha_0}\right) = w \Rightarrow \frac{1}{\alpha} = \frac{1}{\alpha_0} + \beta_2 w \Rightarrow$$

$$\Rightarrow \alpha = \rho^{-1}(w) = \frac{1}{\frac{1}{\alpha_0} + \beta_2 w}$$

$$\Rightarrow \alpha(Q^2) = \rho^{-1}\left(\ln\frac{Q^2}{\mu^2} + \rho(\alpha_\mu)\right) = \frac{1}{\frac{1}{\alpha_0} + \beta_2\left(\ln\frac{Q^2}{\mu^2} + \rho(\alpha_\mu)\right)}$$

$$= \frac{1}{\cancel{\frac{1}{\alpha_0}} + \beta_2\left(\ln\frac{Q^2}{\mu^2} + \frac{1}{\beta_2}\left(\frac{1}{\alpha_\mu} - \cancel{\frac{1}{\alpha_0}}\right)\right)} = \frac{1}{\frac{1}{\alpha_\mu} + \beta_2 \ln\frac{Q^2}{\mu^2}}$$

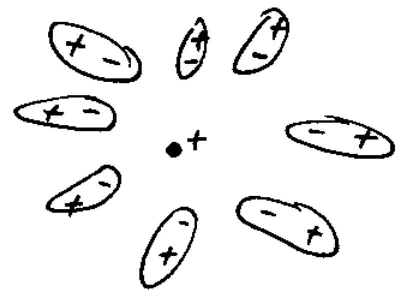
α_0 cancels – not important

$$\Rightarrow \boxed{\alpha(Q^2) = \frac{\alpha_\mu}{1 + \alpha_\mu \beta_2 \ln\frac{Q^2}{\mu^2}}}$$

1-loop running coupling in a gauge theory.

$\Rightarrow$ one can think of running coupling as of the virtual $q\bar{q}$ (or gg) pairs popping out of the vacuum & screening the color charge:

like molecules in a dielectric:



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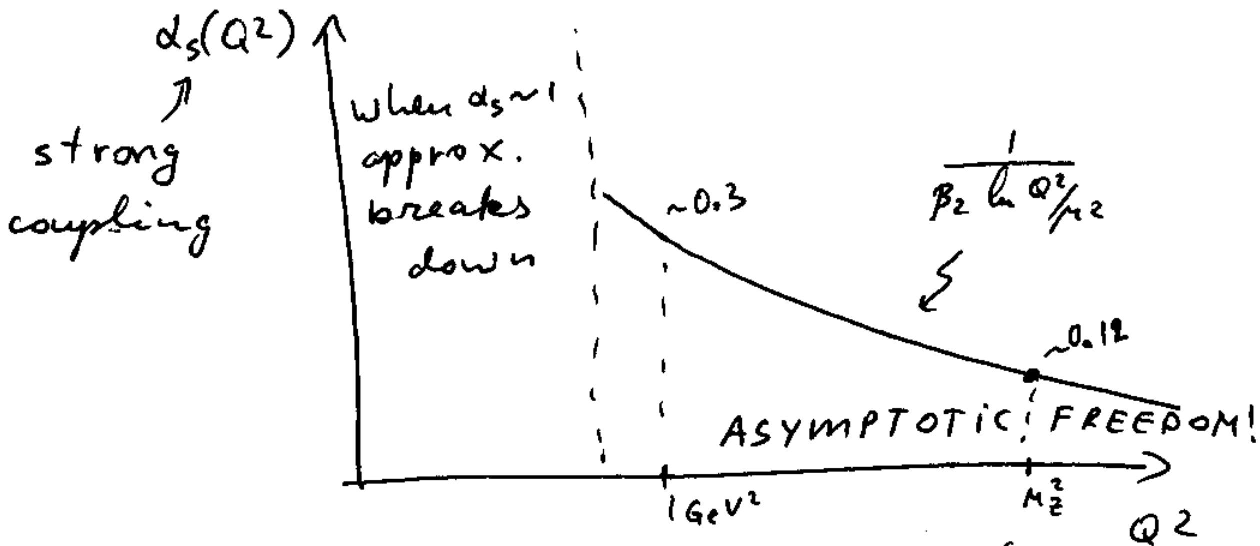
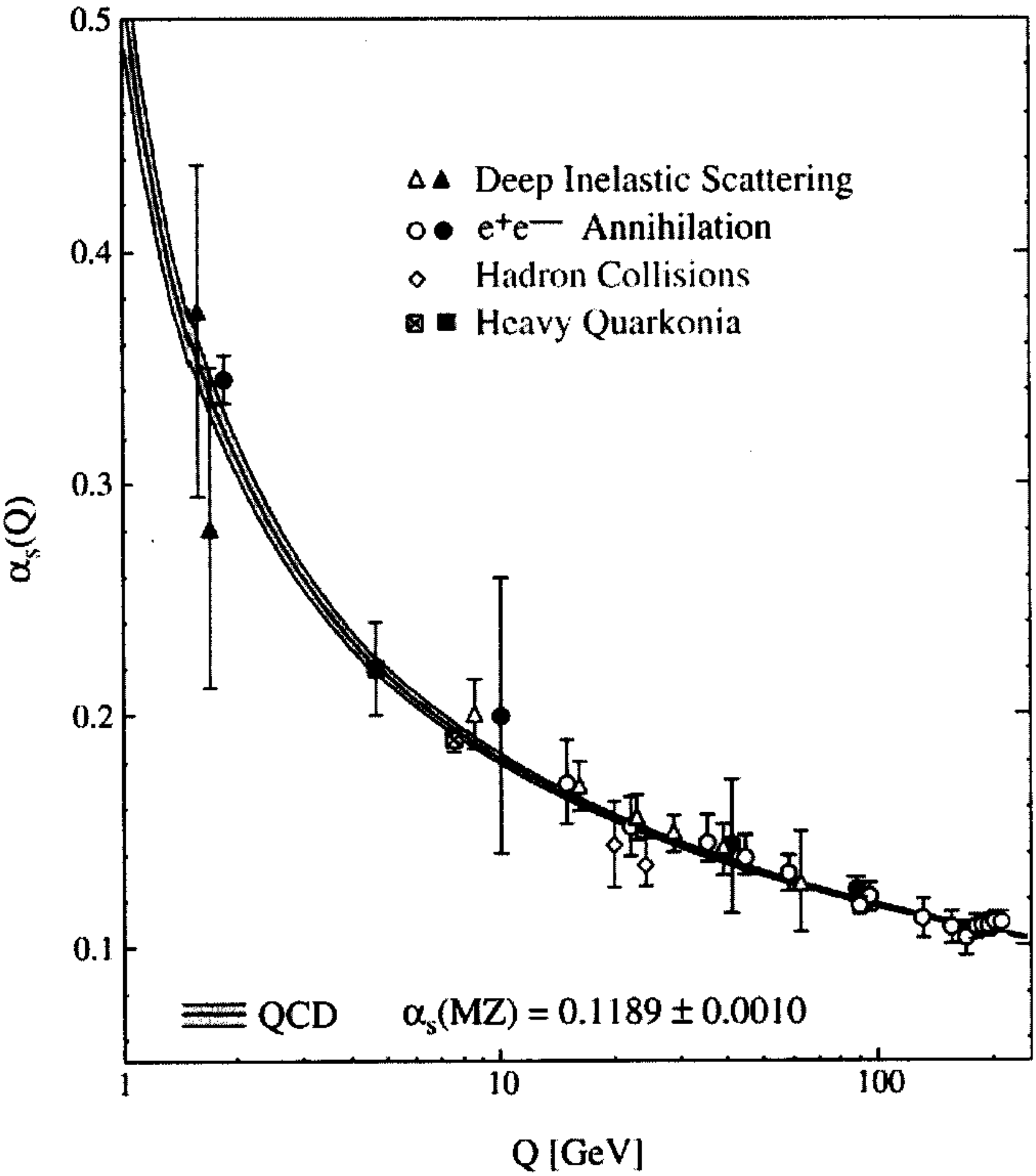
$\Rightarrow$ in QCD $\beta_2 > 0 \Rightarrow$



$\Rightarrow$ at large Q^2 / short distances $(\sim \hbar/Q \sim 1/Q)$ the coupling is small!

$\Rightarrow$ QCD at short distances is weakly coupled ~ quarks and gluons are <u>asymptotically free</u>! (Politzer, Gross, Wilczek '73)

(see attached plot)



0.5
0.4
0.3
0.2
0.1
$\alpha_s(Q)$
Deep Inelastic Scattering
e^+e^- Annihilation
Hadron Collisions
Heavy Quarkonia
QCD $\alpha_s(MZ) = 0.1189 \pm 0.0010$
1
10
100
Q [GeV]

$\Rightarrow$ at large distances / small Q^2 the coupling gets large $\Rightarrow$ pert. th'y breaks down, no one knows what $\alpha_s(Q^2)$ is there.

$\Rightarrow$ when does this happen? write

$$\alpha_s(Q^2) = \frac{\alpha_\mu}{1+\alpha_\mu \beta_2 \ln \frac{Q^2}{\mu^2}} = \frac{1}{\beta_2 \ln \frac{Q^2}{\Lambda^2} + \underbrace{\frac{1}{\alpha_\mu} - \beta_2 \ln \frac{\mu^2}{\Lambda^2}}_{= 0}}$$

define the scale Λ by requiring

$$\Rightarrow \frac{1}{\alpha_\mu} = \beta_2 \ln \frac{\mu^2}{\Lambda^2} \Rightarrow \Lambda^2 = \mu^2 e^{-\frac{1}{\beta_2 \alpha_\mu}} \Rightarrow$$

$\Rightarrow \Lambda^2$ is μ - independent (check).

$$\boxed{\alpha_s(Q^2) = \frac{1}{\beta_2 \ln \frac{Q^2}{\Lambda^2}}}$$

$\Rightarrow$ coupling gets large at $Q^2 \simeq \Lambda^2$.

$\Rightarrow \Lambda^2$ is the fundamental parameter in QCD, usually denoted Λ^2_{QCD}.

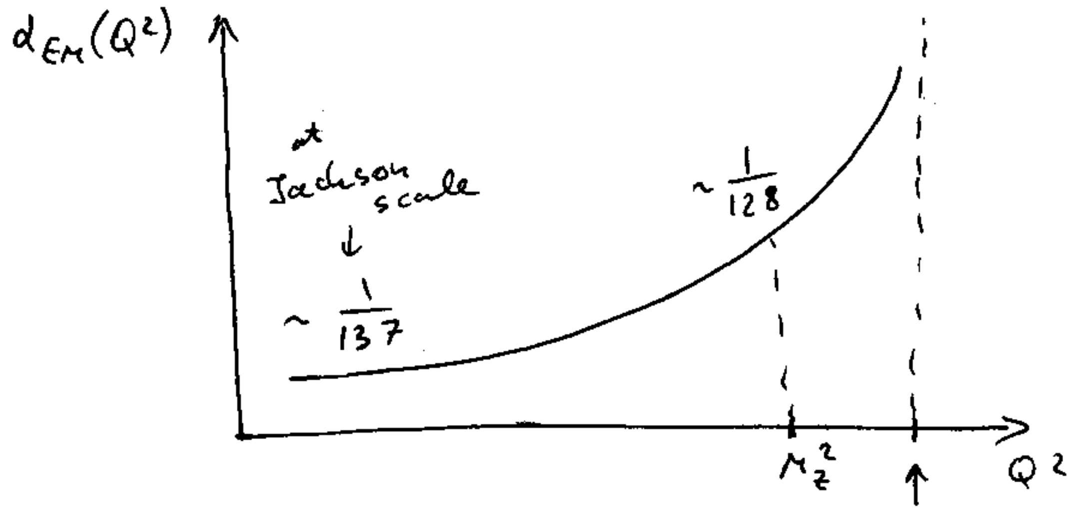
$\Lambda_{QCD} \simeq 200$ MeV (depends on scale)

(Landau pole: $\alpha_s(\Lambda^2) = \infty \Rightarrow$ Landau thought the theory is inconsistent)

II in QED $\beta_2^{QED} < 0 \Rightarrow$

$$\alpha_{EM}(Q^2) = \frac{\alpha_{EM\,\mu}}{1 + \alpha_{EM\,\mu}\underbrace{\beta_2^{QED}}_{= -\frac{1}{3\pi}} \ln \frac{Q^2}{\mu^2}} = \frac{\alpha_\mu}{1 - \frac{\alpha_\mu}{3\pi} \ln \frac{Q^2}{\mu^2}}$$

$$\Rightarrow \alpha_{EM}(Q^2) = \frac{\alpha_\mu}{1 + \frac{\alpha_\mu}{3\pi} \ln \frac{\mu^2}{Q^2}}$$ ~ increases with Q^2



$\Rightarrow$ no asymptotic freedom in QED!

$\Rightarrow$ also has a Landau pole, but at large momenta ~ there QED may map onto some more "fundamental" theory, eliminating Landau pole...